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Three-point functions of quarter BPS operators in $\mathcal{N} = 4$ SYM

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Abstract: In a recent paper [1], $\frac{1}{4}$-BPS chiral primaries were constructed in the fully interacting four dimensional $\mathcal{N} = 4$ Super-Yang-Mills theory with gauge group SU(N). These operators are annihilated by four supercharges, and at order g^2 have protected scaling dimension and normalization. Here, we compute three-point functions involving these $\frac{1}{4}$-BPS operators along with $\frac{1}{2}$-BPS operators. The combinatorics of the problem is rather involved, and we consider the following special cases: (1) correlators $\langle \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\mathrm{BPS}} \rangle$ of two $\frac{1}{2}$-BPS primaries with an arbitrary chiral primary; (2) certain classes of $\langle \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \rangle$ and $\langle \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \rangle$ three-point functions; (3) three-point functions involving the $\Delta \leq 7$ operators found in [1]; (4) $\langle \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \rangle$ correlators with the special $\mathcal{O}_{\frac{1}{4}}$ made of single and double trace operators only. The analysis in cases (1)–(3) is valid for general N, while (4) is a large-N approximation. Order g^2 corrections to all three-point functions considered in this paper are found to vanish.
In the AdS/CFT correspondence, $\frac{1}{4}$-BPS chiral primaries are dual to threshold bound states of elementary supergravity excitations. We present a supergravity discussion of two- and three-point correlators involving these bound states.

Keywords: AdS-CFT Correspondence, Conformal Field Models in String Theory, Conformal and W Symmetry.

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Contents

1. Introduction

The AdS/CFT correspondence [2, 3, 4] provides a powerful tool for deriving dynamical information in $\mathcal{N}$=4 superconformal YM theory outside the regime of weak coupling perturbation theory. Comparison of weak and strong coupling behaviors has given rise to a number of surprising new conjectures [5]–[10] that 2- and 3-point functions, as well as certain "extremal" n-point correlators of $\frac{1}{2}$-BPS operators are "unrenormalized", [11], i.e. when properly defined, their form is independent of the gauge coupling g. $\mathcal{N}$=2 superspace methods have since further confirmed the validity of these results [12, 13, 14]. Extensions to "near-extremal correlators" have unveiled a hierarchical pattern of g-dependence that further generalizes the non-renormalization conjectures in exciting ways [15].

The AdS/CFT correspondence relies on a direct duality between the $\frac{1}{2}$-BPS operators and the canonical fields and states of type-IIB supergravity on $AdS_5 \times S^5$, since they both belong to the shortest multiplets of the superconformal group $\mathrm{SU}(2,2|4)$, [2, 3, 4, 16]. However, in addition to $\frac{1}{2}$-BPS operators, also $\frac{1}{4}$- and $\frac{1}{8}$-BPS operators enjoy certain non-renormalization properties, such as the fact that their scaling dimension is fixed entirely by their internal quantum numbers [17, 18, 19, 1]. The $\frac{1}{4}$- and $\frac{1}{8}$-BPS operators are dual on the AdS side to threshold bound states of elementary supergravity excitations, typically consisting of at least two and three supergravity states respectively, which have not been explored from the AdS point of view.

Through the study of four point functions of $\frac{1}{2}$-BPS operators, certain couplings, such as those of two $\frac{1}{2}$-BPS operators and one $\frac{1}{4}$-BPS operator have been analyzed in SYM theory. In weak-coupling perturbation theory this was done in [10], while arguments based on $\mathcal{N} = 2$ superfield methods were presented in [20]. Furthermore, using the $\mathcal{N} = 4$ superfield approach, a general study of 3-point functions and their non-renormalization properties was initiated in [21]. $\mathcal{N} = 4$ superfield methods, however, require on-shell superfields whose use in the study of off-shell correlators is not fully understood.

In a previous paper [1], a construction was presented for $\frac{1}{4}$-BPS chiral primaries in the fully interacting $\mathcal{N}$=4 SYM theory. In general, these operators are linear combinations of all local, polynomial, gauge invariant, scalar composite operators in the $[p, q, p]$ representations of the R-symmetry group SU(4). Besides the double trace operators from the classification of [18, 19], single trace and other multiple trace operators made of the same scalar fields also have to be taken into account. The coefficients with which they all combine into operators with a well defined scaling dimension are quite involved. However, in the large-N limit, $\frac{1}{4}$-BPS primaries of a special form (those made of the single trace operator and the double trace operator from the classification of [19]) become surprisingly simple. The $\frac{1}{4}$-BPS chiral primaries, like the $\frac{1}{2}$-BPS operators extensively studied in the literature, also have protected two-point functions, at least at order g^2 [1].

Presently, we investigate the (non-) renormalization properties of three-point correlators involving $\frac{1}{4}$-BPS operators along with $\frac{1}{2}$-BPS operators. Given the elaborate combinatorics of the problem, we concentrate on the following special cases. First, we discuss several group theoretic simplifications of the combinatorial factors multiplying the Feynman graphs that contribute to three-point functions of chiral primaries. Based on SU(4) group theory and conformal invariance only, we argue that certain classes of such correlators are protected at order g^2, for all N. In particular, this allows us to compute $\mathcal{O}(g^2)$ corrections to correlators of the form $\langle \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\mathrm{BPS}} \rangle$, where $\mathcal{O}_{\frac{1}{2}}$ are two $\frac{1}{2}$-BPS operators, and $\mathcal{O}_{\mathrm{BPS}}$ is an arbitrary ($\frac{1}{2}$-, $\frac{1}{4}$-, or $\frac{1}{8}$-BPS) chiral primary. Next, we look at the three-point functions $\langle \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \rangle$ and $\langle \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \rangle$, also for general N, where $\mathcal{O}_{\frac{1}{4}}$ are the $\Delta \leq 7$ $\frac{1}{4}$-BPS primaries found in [1]. Then, we carry out a large-N analysis of $\langle \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \rangle$ correlators involving the special $\frac{1}{4}$-BPS operators (mixtures of single and double trace scalar composite operators), for arbitrary Δ. Also in the large-N limit, we identify the corresponding objects in the supergravity description, and compute the correlators on the AdS side of the correspondence.

Finally, we make some speculations. Based on the broad range of special cases studied in this paper and in [1], we conjecture that two- and three-point functions of $\frac{1}{2}$- and $\frac{1}{4}$-operators receive no quantum corrections, for arbitrary N. Additionally, a set of group theoretic considerations of this paper extends straightforwardly from three-point functions to extremal correlators. Therefore, we suggest that extremal correlators involving $\frac{1}{2}$- and $\frac{1}{4}$-operators are protected as well.

2. The fields and the $\mathcal{N} = 4$ SYM lagrangian

Four dimensional $\mathcal{N}$=4 superconformal Yang-Mills theory has been studied extensively for a long time. We begin by reviewing some well known facts.

The dynamical fields in the theory are six real scalars $\phi^I = t^a \phi^I_a$ ($I = 1, \ldots, 6$, $a = 1, \ldots, N^2 - 1$); four Weyl fermions λ_i, $i = 1, \ldots, 4$; and a gauge boson A_μ. All of them are in the adjoint representation of the gauge group SU(N). Under the global bosonic R-symmetry group SU(4) $\sim$ SO(6) of the theory, the scalars form a [0,1,0] = **6**; the fermions are in the [0,0,1] = **4**; and the gauge boson is a singlet.

There are several equivalent ways to formulate the theory, yet none of them show all the features explicitly. For our purposes, it is sufficient to use the component fields of the $\mathcal{N}$=1 superfield formulation of the theory. The manifestly realized part of the R-symmetry group is then SU(3) $\times$ U(1), and the fields are projected accordingly: we have complex scalars z_i and $\bar{z}_i$, $\phi^i = \frac{1}{\sqrt{2}}(z_i + \bar{z}_i)$, $\phi^{i+3} = \frac{1}{i\sqrt{2}}(z_i - \bar{z}_i)$, $i = 1, 2, 3$; and the fermions are written as $\lambda = \lambda_4$, $\psi_i = \lambda_i$, $i = 1, 2, 3$. The classical (euclidean signature) lagrangian can be written as [6]

$$\begin{aligned} L = \mathrm{tr}\, &\left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda + \overline{D_\mu z_j} D^\mu z_j + \frac{1}{2} \bar{\psi}^j \gamma^\mu D_\mu \psi^j \right\} + \\ &+ i\sqrt{2} g f^{abc} \left(\bar{\lambda}_a \bar{z}_b^j L \psi_c^j - \bar{\psi}_a^j R z_b^j \lambda_c \right) - \frac{1}{2} Y f^{abc} \epsilon_{ijk} \left(\bar{\psi}_a^i z_b^j L \psi_c^k - \bar{\psi}_a^i R \bar{z}_b^j \psi_c^k \right) - \\ &- \frac{1}{2} g^2 (f^{abc} \bar{z}_b^j z_c^j)(f^{ade} \bar{z}_d^k z_e^k) + \frac{1}{4} Y^2 f^{abc} f^{ade} \epsilon_{ijk} \epsilon_{ilm} z_b^j z_c^k \bar{z}_d^l \bar{z}_e^m \,, \end{aligned} \tag{2.1}$$

where L and R are chirality projectors (Weyl fermions are equivalent to Majorana in four dimensions). Separate coupling constants g and Y are used to distinguish the terms coming from the gauge and superpotential sectors. The terms in the lagrangian (2.1) proportional to g and g^2 come from the D-terms; while the ones appearing with Y and Y^2, from the F-terms. When $Y = g\sqrt{2}$, SUSY is enhanced from $\mathcal{N}$=1 to $\mathcal{N}$=4; and the R-symmetry group becomes SU(4), although its manifestly realized part is still SU(3)$\times$U(1). Order g^2 calculations are simpler in terms of the fields $\{\lambda, z_i, \bar{z}_i, \psi_i, \bar{\psi}_i \,|\, i = 1, 2, 3\}$, rather than $\{\phi^I, \lambda^j \,|\, I = 1, \ldots, 6,\ j = 1, \ldots, 4\}$, and we will use the lagrangian (2.1) throughout this paper.

3. The operators

We begin setting the stage for computing three-point functions, by describing the operators we will deal with. The construction of gauge invariant scalar composite operators was explained in [1], so here we will briefly review the main points of that discussion, as well as some well established facts.

Four dimensional $\mathcal{N}$=4 SYM is a superconformal theory, and has a global SU(2,2|4) superconformal symmetry group. Operators in the theory fall into multiplets of SU(2,2|4) [17, 18], and chiral primary operators[1] are classified by its maximal bosonic subgroup [19], which includes the R-symmetry group SU(4) $\sim$ SO(6). Chiral primaries are scalar composite operators, and are gauge invariant polynomials in the six real scalars ϕ^I. The most widely studied operators in the theory are the $\frac{1}{2}$-BPS primaries, i.e. chiral primary operators annihilated by 8 out of 16 Poincaré supercharges. These are totally symmetric rank q tensors of the flavor SO(6), with highest weight operators of the form tr $(\phi^1)^q$, minus SO(6) traces. The SU(4) labels of these representations are $[0, q, 0]$, and the SO(6) Young tableau[2] corresponding to totally symmetric rank q tensor is $\boxed{}\!\boxed{\cdots}\!\boxed{}$, namely one row of length q. The $\frac{1}{2}$-BPS chiral primaries are Lorentz scalars, and their conformal dimension is related to their flavor quantum numbers by $\Delta = q$.

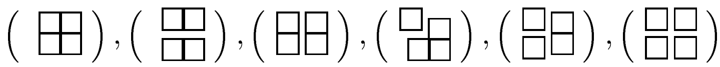
[1] A superconformal primary operator in a given irreducible representation of SU(2,2|4) is the operator of lowest dimension. As a multiplet of SU(4), this operator is unique. Descendant operators in the same irreducible representation are obtained from the primary by application of superconformal generators (i.e. every operator in the representation other than the primary is a descendant). We define a chiral multiplet (or representation) of SU(2,2|4) (also referred to as a BPS, or shortened multiplet) to be such that its superconformal primary operator is annihilated by at least one of the Poincaré supercharges.

[2] See for example [22] for a discussion of irreducible tensors of SO(n).

$\frac{1}{4}$-BPS chiral primaries belong to $[p,q,p]$ representations with $p \geq 2$.[3] In the SO(6) notation, the highest weight state of $[p,q,p]$ corresponds to the

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline 2 & \cdots & 2 & \multicolumn{3}{c}{q} \\ \cline{1-3} \multicolumn{3}{c}{p} \end{array} \tag{3.1}$$

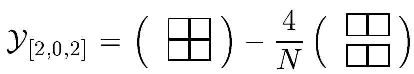
Young tableau. In the free theory, $\frac{1}{4}$-BPS primaries with the highest SU(4) weights are of the form $\mathrm{tr}\,(\phi^1)^{p+q}\,\mathrm{tr}\,(\phi^2)^p$ (modulo (ϕ^1,ϕ^2) antisymmetrizations, and subtraction of the SO(6) traces). However, there are many other ways to partition a given Young tableau. Each partition may result in a different operator after we take the SU(N) traces. To be more explicit, consider the simplest example of $[p,q,p]$ scalar composite operators, namely [2,0,2]. The ways to partition the Young tableaux corresponding to this representation are

 (3.2)

where each continuous group of boxes corresponds to a single SU(N) trace. After taking traces, the last three partitions vanish identically (since $\mathrm{tr}\,\phi^I = 0$); and the "4=2+2" partitions turn out to give the same operator. For a general $[p,q,p]$ representation the arguments are similar, although they become progressively more tedious as $2p+q$ gets larger. Scalar composite operators with $2p+q \leq 7$ are listed in appendix A.

Sometimes, the way we take the SU(N) traces to obtain a $[p,q,p]$ scalar composite operator is not important, and the only relevant information is what fields are used, and that the operator is actually gauge invariant. In such cases, we shall use the notation "[...]" to denote gauge invariant combinations of the fields in brackets. For example, operators corresponding to the highest weight state of representations $[p,q,p]$ will be written as $[(\phi^1)^{(p+q)}(\phi^2)^p] - \mathrm{SO}(6)$ traces, etc.

In the interacting theory, none of the operators obtained by simply partitioning Young tableaux are eigenstates of the dilatation operator (or pure) [1]. Instead, they are mixtures of operators with different scaling dimensions. Proper $\frac{1}{4}$-BPS primaries are certain linear combinations of these mixtures. To find the correct linear combinations, one can look at two-point functions. The $\frac{1}{4}$-BPS primaries are identified as the operators which receive no $\mathcal{O}(g^2)$ corrections to two-point functions among themselves or with other scalar composite operators. The protected scaling dimension of a $[p,q,p]$ chiral primary is $\Delta = 2p+q$. The simplest example of a $\frac{1}{4}$-BPS primary is the [2,0,2] operator



$$= 2\left[\mathrm{tr}\,z_1^2\,\mathrm{tr}\,z_2^2 - (\mathrm{tr}\,z_1 z_2)^2\right] - \frac{4}{N}\left[\mathrm{tr}\,z_1^2 z_2^2 - \mathrm{tr}\,(z_1 z_2)^2\right] \tag{3.3}$$

which has been previously studied in [10] and [1].

4. Contributing diagrams

Next, we sort out the diagrams which contribute to correlators of these operators at order g^0 and g^2.

[3]They are $\frac{1}{2}$-BPS for $p = 0$; and vanish for $p = 1$ after we take the SU(N) traces.

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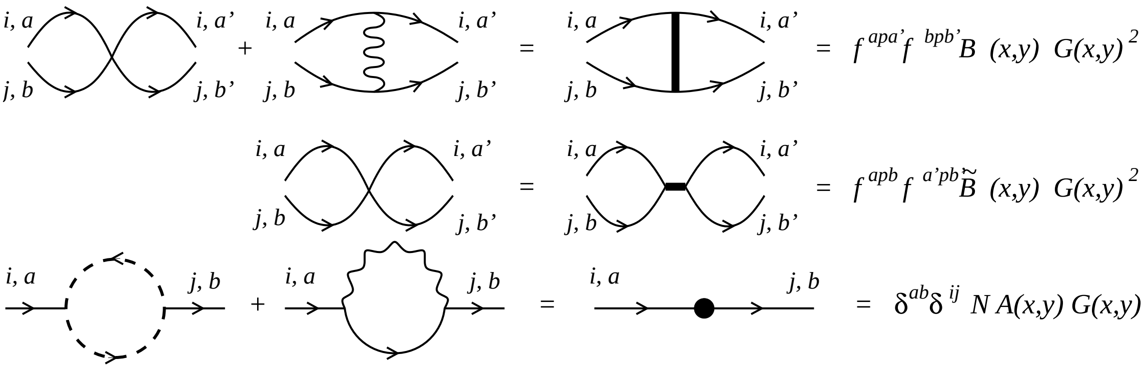


Figure 1: Structures contributing to two-point functions of scalars at order g^2 through four-scalar blocks and the propagator. Thick lines correspond to exchanges of the gauge boson, and of the auxiliary fields F_i and D (in the $\mathcal{N}$=1 formulation; after integrating out F_i and D, the $zz\bar{z}\bar{z}$ vertex). The scalar propagator remains diagonal in both color and flavor indices at order g^2. At order g^2, there are corrections to the scalar propagator coming from a fermion loop (dashed line) and a gauge boson semi-loop (wiggly line). Also, blocks involving four scalars get contributions from a single gauge boson exchange, and from the four-scalar vertex.

The two-point functions of the scalar composites discussed in [1], have the schematic form

$$\left\langle \left[{z_1}^{(p+q)} {z_2}^{p} \right](x) \left[\bar{z}_1^{(p+q)} \bar{z}_2^{p} \right](y) \right\rangle , \tag{4.1}$$

where $[\ldots]$ are some gauge invariant combinations. The free field part of (4.1) is given by a power of the free correlator $[G(x,y)]^{(2p+q)}$, times a combinatoric factor. From the lagrangian (2.1) we can read off the structures for the leading correction to the propagator, and the four-scalar blocks. These corrections are shown in figure 1, where they are categorized according to their gauge group (color) index structure (we use the same notation as in [6]). The scalar propagator remains diagonal in both color and flavor indices at order g^2. Gauge fixing and ghost terms in the lagrangian do not contribute at this order (as we are only looking at operators made up of scalars).

Three-point functions to be considered in this paper are of the form

$$\left\langle \left[z^{k+l} \right](x) \left[\bar{z}^{k+m} \right](y) \left[\bar{z}^{l} z^{m} \right](w) \right\rangle . \tag{4.2}$$

The free result is just the product of appropriate powers of free correlators $[G(x,y)]^k \times [G(x,w)]^l [G(y,w)]^m$. The same structures that contribute to the two-point functions at order g^2 (see figure 1), also contribute to the three-point functions (4.2). Apart from these, there are new building blocks, shown in figure 2. They have the same index structure, but are now functions of three space-time coordinates rather than two.

Notice that the F-term corrections proportional to $\tilde{B}(x,y)$ in figure 1, and the last graph (proportional to $\tilde{C}(x;y,w)$) in figure 2, are antisymmetric in i and j, hence they are absent when the scalars in the four legs have pairwise the same flavor. For the same reason, these corrections are also absent when the operator at point x is symmetric in all of its flavor indices. In particular, this is the case when the operator at x is $\frac{1}{2}$-BPS.

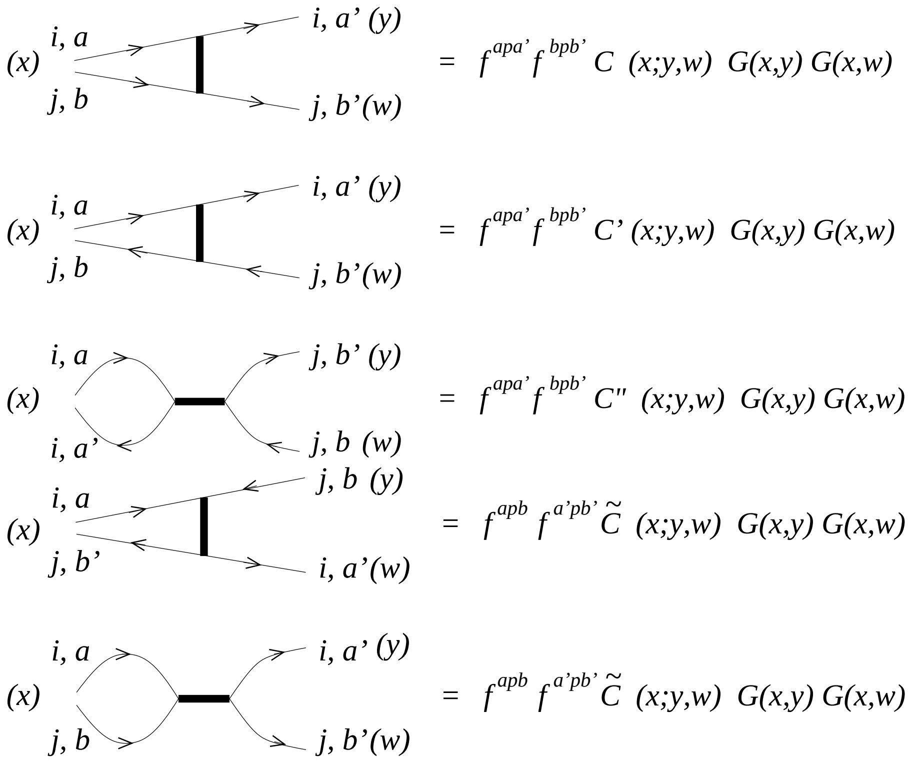


Figure 2: Building blocks for g^2 corrections to three-point functions. The three space-time points are x (with two legs attached) and y and w (with a single leg each).

5. Restrictions from $\mathcal{N} = 4$ SUSY and gauge invariance

The form of quantum corrections to two and three-point functions is known [6]. Space-time coordinate dependence of the Feynman diagrams contributing to these correlators at order $\mathcal{O}(g^2)$ is constrained, since all exchanged fields are massless. We know the parametric form of the functions $A(x,y)$, $B(x,y)$, $\tilde{B}(x,y)$; and $C(x;y,w)$, $C'(x;y,w)$, $C''(x;y,w)$, and $\tilde{C}(x;y,w)$, without having to perform integrals explicitly. Functions which depend on two space-time points, are of the form $A(x_1,x_2) = a \log x_{12}^2\mu^2 + b$ with $x_{ij} \equiv x_i - x_j$; three-point contributions look like $C(x_1;x_2,x_3) = a' \log x_{12}^2 x_{13}^2 \mu^4 - a'' \log x_{23}^2 + b'$ (making use of the $x_2 \leftrightarrow x_3$ symmetry of these building blocks).

$\mathcal{N}$=4 SUSY tells us more. From non-renormalization of two and three-point functions of operators in the stress tensor multiplet, one can see [6] that $B(x,y) = -2A(x,y)$, and $C'(0;x,y) + \tilde{C}(0;x,y) = -C(0;x,y)$; the authors of [6] chose to combine these and call it just C'.[4] The coefficients a', a'' and b' are determined[5] in terms of a and b:

$$\begin{aligned} A(x,0) &= -\frac{1}{2}B(x,0) = a \log x^2\mu^2 + b \\ -C(0;x,y) &= a \log \frac{x^2 y^2 \mu^2}{(x-y)^2} + b\,. \end{aligned} \tag{5.1}$$

[4] One way to see this is to consider the protected correlators of [0,2,0] scalar composite operators $\langle \mathrm{tr}\, z_1 z_2(x)\ \mathrm{tr}\, \bar{z}_1\bar{z}_2(y)\rangle$, and $\langle [z^2](x)[\bar{z}^2](y)[z\bar{z}](w)\rangle$ and $\langle [z\bar{z}](x)[z\bar{z}](y)[z\bar{z}](w)\rangle$.

[5] This follows from $C(x;y,w) + C(y;x,w) + C(w;x,y) + A(x,y) + A(y,w) + A(x,w) = 0$.

Therefore, the net contribution to the three-point function (4.2) of the $\mathcal{O}(g^2)$ diagrams involving a gauge boson exchange (the ones proportional to A, B, and C), is

$$\begin{aligned}&\langle [z^k](x_1)[\bar{z}^l](x_2)[z^m\bar{z}^n](x_3)\rangle|_{(A+B+C)} = \\ &\quad = a(c_g^{12}\log x_{12}^2\mu^2 + c_g^{13}\log x_{13}^2\mu^2 + c_g^{23}\log x_{23}^2\mu^2) + bc_g^{123}\,,\end{aligned} \tag{5.2}$$

where and c_g^{ij} and c_g^{123} are some combinatorial coefficients.

Now we use gauge invariance of the theory. On the one hand, we observe that the coefficients a and b are gauge dependent [1],

$$A(x,0) = \frac{1}{2}\pi^2 g^2 \xi \left[\log x^2\mu^2 + \log 4\pi - \gamma\right] + (\xi\text{-independent})\,, \tag{5.3}$$

where ξ is the gauge fixing parameter. On the other hand, a correlator of gauge invariant operators can not depend on ξ. Therefore, the combinatorial coefficients multiplying a and b in equation (5.2) must vanish, $c_g^{ij} = c_g^{123} = 0$. Hence, the D-term diagrams proportional to A, B, and C all cancel; their net contribution to the three-point functions (4.2) is zero.

So just like in the case of two-point functions, we only have to consider the F-term graphs. They are proportional to $\tilde{B}$ and $\tilde{C}$, the only gauge independent diagrams around ($C' = -(C+\tilde{C})$ and $C'' = C - \tilde{C}$ do not have to be treated separately as they are linear combinations of the other ones).

In the $\mathcal{O}(g^2)$ calculations of correlators of $\frac{1}{2}$-BPS operators [6] and [8], there were no other contributions to three-point functions except for those proportional to A and B. Thus, gauge invariance together with $\mathcal{N}$=4 SUSY (which is needed to relate C and B to A) guarantees that the correlators of [6] and [8] receive no order g^2 corrections.

6. Position dependence of $\tilde{B}$ and $\tilde{C}$

Having shown that D-term corrections to three-point functions (4.2) are absent, it remains to consider the F-term interactions. In this section we derive a relation between functions $\tilde{B}$ and $\tilde{C}$, which will play a key role in the analysis of three-point functions of $\frac{1}{4}$-BPS chiral primaries, see section 8.1.1.

Space-time position dependence of $\tilde{B}$ and $\tilde{C}$ (shown figures 1 and 2) is parametrically determined to be $\tilde{B}(x,0) = \tilde{a}\log(x^2\mu^2) + \tilde{b}$ and $\tilde{C}(0;x,y) = \tilde{a}'\log(x^2y^2\mu^4) - \tilde{a}''\log((x-y)^2\mu^2) + \tilde{c}$. Furthermore, the leading divergent behavior can be read off from the integrals unambiguously, and so from the limit $\tilde{C}(0;x,y\to x)$ we infer $\tilde{a}' = \frac{1}{2}\tilde{a}$.

To evaluate the remaining coefficients $\tilde{a}$, $\tilde{a}''$, and $\tilde{b}$, we use differential regularization [23],[6] or a simpler equivalent prescription: replace $1/x^2 \to 1/(x^2+\epsilon^2)$ for scalar propagators inside integrals. With this prescription

$$\begin{aligned}\tilde{B}(x,0) &= -\frac{1}{4}Y^2\int\frac{(d^4z)\left[4\pi^2x^2\right]^2}{\left[4\pi^2((z-x)^2+\epsilon^2)\right]^2\left[4\pi^2(z^2+\epsilon^2)\right]^2} \\ &= -Y^2\frac{1}{32\pi^2}\left[\log(x^2/\epsilon^2)-1\right]\end{aligned} \tag{6.1}$$

[6]This is the fastest way to calculate the integral for $\tilde{C}$, but one can obtain the same results using dimensional regularization.

is the regularized two-point function, while the three-point function becomes

$$\begin{aligned}\tilde{C}(x;y,0) &= -\frac{1}{4}Y^2\int\frac{(d^4z)\left[4\pi^2x^2\right]\left[4\pi^2(x-y)^2\right]}{\left[4\pi^2((z-x)^2+\epsilon^2)\right]^2\left[4\pi^2((z-y)^2+\epsilon^2)\right]\left[4\pi^2(z^2+\epsilon^2)\right]}\\ &= -Y^2\frac{1}{64\pi^2}\left[\log\frac{x^2(x-y)^2}{y^2\epsilon^2}\right]\end{aligned} \tag{6.2}$$

(The numerators inside the integrals come about because of the powers of free scalar propagator in the definitions of $\tilde{B}$ and $\tilde{C}$, see figures 1 and 2.) Therefore,

$$\tilde{C}(x;y,0)+\tilde{C}(y;x,0)-\tilde{B}(x,y) = -Y^2\times\frac{1}{32\pi^2} \tag{6.3}$$

is a nonzero constant (for $\mathcal{N}$=4 SUSY, $Y^2=2g^2$).

The value of the constant $-Y^2/32\pi^2$ in equation (6.3) does not depend on the regulator ϵ. Also note that with the "point splitting regularization" one would get the incorrect result of vanishing constant in (6.3).

7. Structure of the three-point functions

With the results of section 5 at hand, we can write down the form of a general three-point function of scalar composite operators (4.2) to order g^2:

$$\begin{aligned}\Big\langle\left[z^{k+l}\right](x)\left[\bar{z}^{k+m}\right](y)\left[\bar{z}^l z^m\right](w)\Big\rangle &= G(x,y)^kG(x,w)^lG(w,y)^m\times\\ &\times\Big(\alpha_{\rm free}+\tilde{\beta}_{xy}\tilde{B}(x,y)+\tilde{\beta}_{xw}\tilde{B}(x,w)+\tilde{\beta}_{yw}\tilde{B}(y,w)+\\ &\quad+\tilde{\gamma}_x\tilde{C}(x;y,w)+\tilde{\gamma}_y\tilde{C}(y;x,w)+\tilde{\gamma}_w\tilde{C}(w;x,y)+\\ &\quad+\mathcal{O}(g^4)\Big)\,,\end{aligned} \tag{7.1}$$

where $\alpha_{\rm free}$, $\tilde{\beta}$-s and $\tilde{\gamma}$-s are some combinatorial coefficients. Using the expressions (6.1) and (6.2) from section 6, we can determine the $\mathcal{O}(g^2)$ position dependence of (7.1) completely — if we know these combinatorial coefficients. Together with conformal invariance, and the SU(4) symmetry properties of the operators in (7.1), we can often go a long way to figuring out which of the combinatorial coefficients must vanish, without doing any actual calculations.

7.1 Space-time coordinate dependence

Like in the case of two-point functions, conformal invariance restricts position dependence of three-point correlators of pure operators (i.e. ones which have a well defined scaling dimension). Consider three (gauge invariant Lorentz scalar) operators $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$, of dimensions $\Delta_i = k_i+\delta_i$, inserted at corresponding points x_i. Let k_i be integers; and δ_i, the order g^2 corrections to the scaling dimensions (which may or may not be zero). The three-point function $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$ is completely determined up to a multiplicative constant

$C_{123} = C^0_{123} + C^1_{123}$ (where again C^0_{123} is the free field result and $C^1_{123} \sim g^2$),

$$\begin{aligned}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle &= \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}} \\ &= \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_{\rm free}\Bigg(1 + \frac{C^1_{123}}{C^0_{123}} - \delta_1 \log\frac{x_{12}^2 x_{13}^2}{x_{23}^2\epsilon^2} - \\ &\qquad - \delta_2 \log\frac{x_{21}^2 x_{23}^2}{x_{13}^2\epsilon^2} \delta_3 \log - \frac{x_{31}^2 x_{32}^2}{x_{12}^2\epsilon^2} + \mathcal{O}(g^4)\Bigg) \end{aligned} \tag{7.2}$$

with $x_{ij} = x_i - x_j$ as usual.

Suppose that all three operators have protected scaling dimensions, $\delta_i = 0$. Then (7.2) reduces to

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_{\rm free}\Big(1 + C^1_{123}/C^0_{123}\Big) \tag{7.3}$$

and no logs arise. In this case, the combinatorial factors in (7.1) satisfy

$$\tilde\gamma_x = -\left(\tilde\beta_{xy} + \tilde\beta_{xw}\right) \tag{7.4}$$

$$\tilde\gamma_y = -\left(\tilde\beta_{xy} + \tilde\beta_{yw}\right) \tag{7.5}$$

$$\tilde\gamma_w = -\left(\tilde\beta_{xw} + \tilde\beta_{yw}\right) \tag{7.6}$$

and we need to calculate only three coefficients (the $\tilde\beta$-s, for example), to find all the $\mathcal{O}(g^2)$ corrections to this correlator. In fact, the only allowed correction is the constant $\tilde\beta \equiv \tilde\beta_{xy} + \tilde\beta_{xw} + \tilde\beta_{yw}$ times $(-Y^2/32\pi^2)$. To show that a three-point function of BPS operators is protected, we have to demonstrate that $\tilde\beta = 0$.

7.2 Group theory simplifications

There are several simplifications which set some of the combinatorial coefficients in (7.1) to zero. These considerations are based on the underlying SU(4) $\sim$ SO(6) symmetry of the theory only, and are applicable for general N. We will leave aside the trivial case when the correlator is forced to vanish by group theory, and assume that the Born level coefficient in (7.1) $\alpha_{\rm free} \neq 0$.

The simplest BPS operators are $\frac{1}{2}$-BPS chiral primaries, gauge invariant scalar composites in $[0,q,0]$ representations of SO(6). These are totally symmetric tensors of SO(6), so if for example the operator $\mathcal{O}_x$ is $\frac{1}{2}$-BPS, the coefficients $\tilde\beta_{xy} = \tilde\beta_{xw} = \tilde\gamma_x = 0$ since the diagrams they multiply are antisymmetric in the flavor indices of $\mathcal{O}_x$.[7] Similarly, if both $\mathcal{O}_x$ and $\mathcal{O}_y$ are $\frac{1}{2}$-BPS and $\mathcal{O}_w$ is any BPS operator, we have $\tilde\beta_{xy} = \tilde\beta_{xw} = \tilde\gamma_x = \tilde\beta_{yw} = \tilde\gamma_y = 0$, and hence $\tilde\beta = \tilde\beta_{xy} + \tilde\beta_{xw} + \tilde\beta_{yw} = 0$; there are no $\mathcal{O}(g^2)$ corrections in this case. In particular, this reproduces the result of [6] when all three $\mathcal{O}_{x,y,w}$ are $\frac{1}{2}$-BPS chiral primaries.

[7] If we chose $\mathcal{O}_w$ as such a $\frac{1}{2}$-BPS operator, we would not be able to conclude that $\tilde\gamma_w = 0$ just from the symmetries of $\mathcal{O}_w$: the fourth diagram of figure 2, is not antisymmetric in flavor indices at the vertex where the operator is made of both z-s and $\bar z$-s. However, using equation (7.6) we find $\tilde\gamma_w = \tilde\beta_{xw} + \tilde\beta_{yw} = 0$ since $\tilde\beta_{xw} = \tilde\beta_{yw} = 0$ when all three operators have protected scaling dimensions.

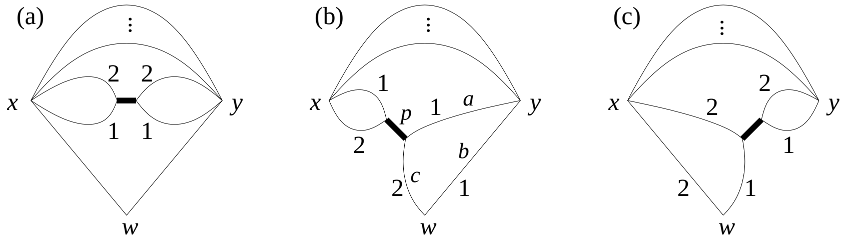


Figure 3: F-term contributions to $\langle \mathcal{O}_{\rm BPS}(x)\mathcal{O}'_{\rm BPS}(y)\mathcal{O}_{\frac{1}{2}}(w)\rangle_{g^2}$ in the case when the correlator can be "partitioned into two flavors:" (a) proportional to $\tilde{B}(x,y)$; (b) proportional to $\tilde{C}(x;y,w)$; (c) proportional to $\tilde{C}(y;x,w)$.

The "standard" way to choose SU(4) weights of operators in a three point function is for one of them to be a highest weight state λ in its representation; another operator to be the lowest weight state $-\lambda'$ in its; and the third operator to have weight $\lambda' - \lambda$. But in some cases, the combinatorics simplifies if we chose the weights differently, as will be illustrated below.

Suppose that $\mathcal{O}_w$ is $\frac{1}{2}$-BPS, and furthermore we can "partition the correlator into two flavors," i.e. choose the operators such that $\mathcal{O}_w = [\bar{z}_1^m z_2^n]$ while $\mathcal{O}_x = [z_1^k z_2^l]$ and $\mathcal{O}_y = [\bar{z}_1^{(k-m)}\bar{z}_2^{(l-n)}]$. Consider a diagram proportional to $\tilde{C}(x;y,w)$, see figure 3(b). The sum of all such diagrams is symmetric in the color indices a and b at y (since the scalars at y used in this diagram have the same flavors, z_1^a and z_1^b), and symmetric in colors at w (since $\mathcal{O}_w$ is $\frac{1}{2}$-BPS, it is symmetric in the indices b and c). But it must be antisymmetric in the color indices a and c of the z_1^a and z_2^c leaving the interaction vertex (as this diagram comes with a factor of f^{apc}, see figure 1). We conclude that all such diagrams cancel, and so $\tilde{\gamma}_x = 0$. In the same fashion, we conclude that $\tilde{\gamma}_y = 0$ as well, and together with $\tilde{\beta}_{yw} = \tilde{\beta}_{xw} = 0$ (as $\mathcal{O}_w$ is $\frac{1}{2}$-BPS), we find that $\beta = 0$ when $\mathcal{O}_x$ and $\mathcal{O}_y$ are any operators with protected scaling dimensions.

There is another type of three-point functions of BPS chiral primaries which receive no $\mathcal{O}(g^2)$ corrections by similar considerations. Consider a correlator (7.1) such that $\mathcal{O}_x$ is made of z_1 and z_2; $\mathcal{O}_y$ made of $\bar{z}_1$ and $\bar{z}_3$; and $\mathcal{O}_w$, made of $\bar{z}_2$ and z_3, i.e. a correlator of the form

$$\langle [z_1^m z_2^n](x)\ [\bar{z}_1^m \bar{z}_3^k](y)\ [z_3^k \bar{z}_2^n](w)\rangle\ . \tag{7.7}$$

This correlator is "partitioned into three disjoint flavors." Order g^2 contributions to this three-point function are shown in figure 4. There are no corrections proportional to any $\tilde{B}$-s since all lines within any rainbow carry the same flavor, so immediately $\tilde{\beta}_{xy} = \tilde{\beta}_{yw} = \tilde{\beta}_{xw} = 0$ and hence there are no $\mathcal{O}(g^2)$ corrections here, as well.

Finally, extremal three-point functions can be analyzed in a simple way. Here, the scaling dimension of one of the operators is equal to the sum of scaling dimensions of the other two.[8] Suppose that $\Delta_x + \Delta_y = \Delta_w$ in (7.1). At Born level, there are no $G(x,w)$ propagators, and so there are no corrections proportional to $\tilde{B}(x,y)$, $\tilde{C}(x;y,w)$, or

[8]In general, $(n+1)$-point functions $\langle \mathcal{O}_0(x_0)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle$ are called extremal if one of the scaling dimensions is the sum of all the others, $\Delta_0 = \Delta_1 + \ldots + \Delta_n$.

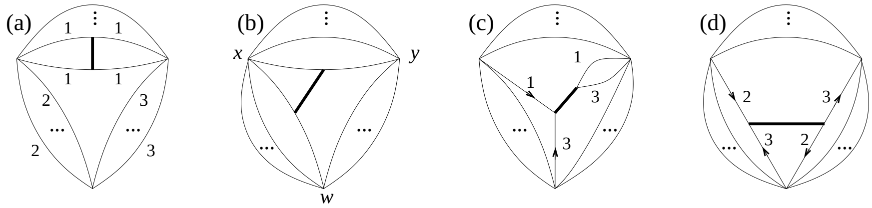


Figure 4: Order g^2 corrections to correlators of the form (7.7): (a) and (b) includes a gauge boson exchange; (c) and (d) F-terms. Self energy contributions (not shown) also include a gauge boson exchange.

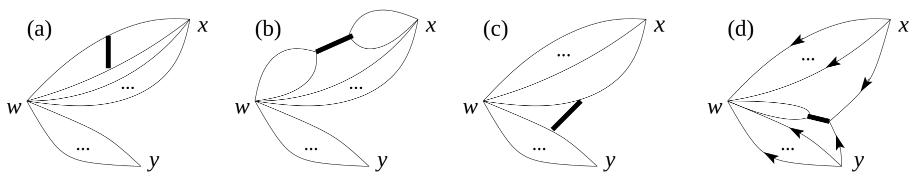


Figure 5: Order g^2 corrections to extremal correlators: (a) and (b) within a single peddle; (c) and (d) between the two peddles. Self energy contributions (not shown) and diagrams (a) and (c) are gauge dependent, while (b) and (d) diagrams arise from the F-terms.

$\tilde{C}(y; x, w)$, see figure 5. Together with the constraints (7.4)–(7.6), this determines $\tilde{\beta} = 0$ when the three (Lorentz scalar) operators inserted x, y, and w are arbitrary operators with protected scaling dimensions.

Another remark about extremal correlators is in order. As it is easy to see, one of the above group theory simplifications generalizes straightforwardly to extremal correlators of chiral primaries. Namely, if all operators except for one are $\frac{1}{2}$-BPS (and the remaining one is an arbitrary chiral primary), extremal correlators receive no order g^2 corrections.

8. Three-point functions of BPS operators

We are now ready to discuss correlators of three BPS chiral primaries. The simplest correlators $\langle \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\frac{1}{2}} \rangle$ (where each $\mathcal{O}_{\frac{1}{2}}$ stands for a $\frac{1}{2}$-BPS operator), were considered by the authors of [6], who found that three-point functions of $\frac{1}{2}$-BPS operators do not get corrected at order g^2, for any N. These are a special case of correlators of the form $\langle \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\frac{1}{2}} \mathcal{O}_{\rm BPS} \rangle$, which we discussed in section 7.2 (here $\mathcal{O}_{\rm BPS}$ is an arbitrary BPS operator). Such three-point functions receive no $\mathcal{O}(g^2)$ corrections by group theory reasoning.

8.1 Correlators $\left\langle \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{2}} \right\rangle$

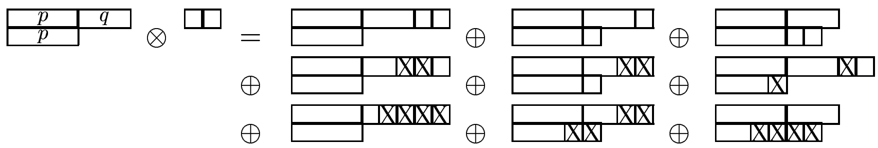
Not all three-point functions of chiral primaries can be simplified using the results of section 7.2, so occasionally we will have to actually compute some of the combinatorial coefficients. In this section we will look at correlators of two $\frac{1}{4}$-BPS operators with one $\frac{1}{2}$-BPS operator.

8.1.1 $\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)(\mathbf{tr}\,\bar{z}tz)(w)\rangle$

The simplest $\langle \mathcal{O}_{\frac{1}{4}}\mathcal{O}_{\frac{1}{4}}\mathcal{O}_{\frac{1}{2}}\rangle$ three-point functions are of the form

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}'(y)(\mathrm{tr}\,X^2)(w)\rangle\,, \tag{8.1}$$

where the $\frac{1}{2}$-BPS primary $\mathrm{tr}\,X^2$ is a scalar composite operator in the [0,2,0] of SU(4). Group theory restricts the quantum numbers of operators which can have nontrivial three point functions. Tensoring $[p,q,p]\otimes[0,2,0]$ using Young diagrams of SO(6) gives



$$\oplus \ \dots \tag{8.2}$$

where in the first row there are no contractions (i.e. SO(6) traces), only symmetrizations and antisymmetrizations; in the second row, one contraction; and in the third row, two contractions; the "…" stands for tableaux with more than two rows.[9] In terms of Dynkin labels, equation (8.2) reads

$$\begin{aligned}
[p,q,p]\otimes[0,2,0] \;=\; & [p,q+2,p]\oplus[p+1,q,p+1]\oplus[p+2,q-2,p+2]\oplus\\
& \oplus[p,q,p]\oplus[p+1,q-2,p+1]\oplus[p-1,q+2,p-1]\oplus\\
& \oplus[p,q-2,p]\oplus[p-1,q,p-1]\oplus[p-2,q+2,p-2]\oplus\\
& \oplus\cdots\,.
\end{aligned} \tag{8.3}$$

Now, the "$\cdots$" stands for representations with $[r,s,r+2k]$ Dynkin labels with $k\neq 0$. Thus the only three-point functions of the form (8.1) which can possibly have a nonzero value are the extremal correlators

$$\begin{aligned}
&\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q-2,p]}(y)(\mathrm{tr}\,\bar{z}_1^2)(w)\rangle\\
&\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p-2,q+2,p-2]}(y)(\mathrm{tr}\,\bar{z}_2^2)(w)\rangle\\
&\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p-1,q,p-1]}(y)(\mathrm{tr}\,\bar{z}_1\bar{z}_2)(w)\rangle
\end{aligned} \tag{8.4}$$

which correspond to those diagrams in (8.2) with zero or maximal number of contractions; and non-extremal correlators

$$\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)(\mathrm{tr}\,\bar{z}tz)(w)\rangle \tag{8.5}$$

$$\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p-1,q+2,p-1]}(y)(\mathrm{tr}\,\bar{z}_2 z_1)(w)\rangle\,, \tag{8.6}$$

where t is a diagonal SU(3) generator. All other correlators of the form (8.1) either vanish because the tensor product of irreps $[p,q,p]$ and $[0,2,0]$ does not contain $[r,s,r]$, or are related to the ones in (8.4)–(8.6).

[9] Tensoring representations in the manner of equation (8.2) gets messy for larger representations. Another method (of Berenstein-Zelevinsky triangles) is discussed in appendix B.

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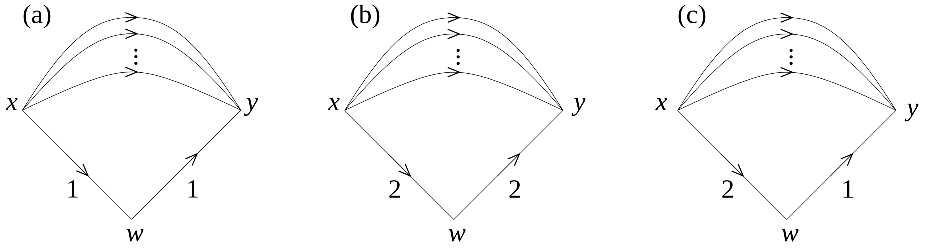


Figure 6: Nonvanishing Born diagrams for non-extremal three-point functions $\langle \mathcal{O}_{[p,q,p]}\bar{\mathcal{O}}_{[p,q,p]}(\mathrm{tr}\,\bar{z}tz)\rangle$ *(a-b)*; $\langle \mathcal{O}_{[p,q,p]}\bar{\mathcal{O}}_{[p-1,q+2,p-1]}(\mathrm{tr}\,\bar{z}_2 z_1)\rangle$ *(c)*.

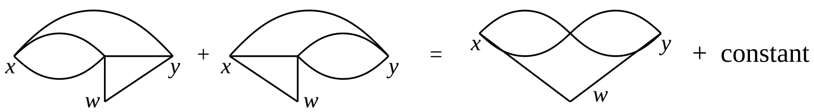
Extremal three-point functions were discussed in section 7.2, and were found to be protected at order g^2. The only correlators of the form $\langle \mathcal{O}\bar{\mathcal{O}}'\mathrm{tr}\,X^2\rangle$ we need to consider are those given by (8.5) and (8.6). However, the three-point functions of figure 6(c), must in fact vanish: $\mathrm{tr}\,\bar{z}_2 z_1 = \frac{1}{2}\bar{z}_2^a z_1^a$ is diagonal in color indices, and hence the combinatorial factors for the Born graph of $\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p-1,q+2,p-1]}(y)(\mathrm{tr}\,\bar{z}_2 z_1)(w)\rangle$ are proportional to the ones for the two-point function $\langle \mathcal{O}_{[p,q,p]}\bar{\mathcal{O}}_{[p-1,q+2,p-1]}\rangle = 0$. The same thing happens at order g^2, etc.[10] So correlators (8.6), although allowed by (8.3), are in fact forbidden by a combination of SU(N) and SU(4) group theory.

Correlators $\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)(\mathrm{tr}\,\bar{z}tz)(w)\rangle$ are the only ones that remain to be considered. The contributing Born level diagrams are shown in figure 6(a,b), and the $\mathcal{O}(g^2)$ graphs appear in figure 7 (corrections to the scalar propagator are not shown, but are also present). Repeating the arguments of [6] from the $\frac{1}{2}$-BPS calculations, we see that the combinatorial structure of this three-point function $\langle \mathcal{O}\bar{\mathcal{O}}(\mathrm{tr}\,\bar{z}tz)\rangle$ is the same as that of the two-point function $\langle \mathcal{O}\bar{\mathcal{O}}\rangle$. At Born level, we find that

$$\begin{aligned}&\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)(\mathrm{tr}\,\bar{z}tz)(w)\rangle|_{\mathrm{free}} = \\ &\qquad = \frac{1}{2}[(p+q)t_{11} + pt_{22}]\left[\frac{G(x,w)G(y,w)}{G(x,y)}\right]\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)\rangle|_{\mathrm{free}}\,. \end{aligned} \tag{8.7}$$

At order g^2, the contributions proportional to $\tilde{B}$ and $\tilde{C}$ (diagrams ($a1$) and ($b1$) in figure 7) have the same index structure, which in turn is identical to that of the two-point functions $\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)\rangle$. Because $\mathrm{tr}\,\bar{z}_1 z_1$ is diagonal in color indices, its only effect on the combinatorics is to distinguish the pair of indices which go to $\mathcal{O}_w$ rather than stretch directly between $\mathcal{O}_x$ and $\mathcal{O}_y$.

There is a curious relation between the functions $\tilde{B}(x,y)$ and $\tilde{C}(x;y,w)$, which can be graphically expressed as[11]



(8.8)

This is a consequence of conformal invariance and nonrenormalization of the scaling dimension of $\mathrm{tr}\,X^2$. To see how this comes about, let $\mathcal{O}$ be an arbitrary (not necessarily

[10] Explicitly, in section 7.2 we saw that the $\mathcal{O}(g^2)$ part of this three-point function vanishes.

[11] The fact that $\tilde{C}(x;y,w) + \tilde{C}(y;x,w) - \tilde{B}(x,y)$ is just a constant was established in section 6 by an explicit calculation. The value of this constant was also found there.

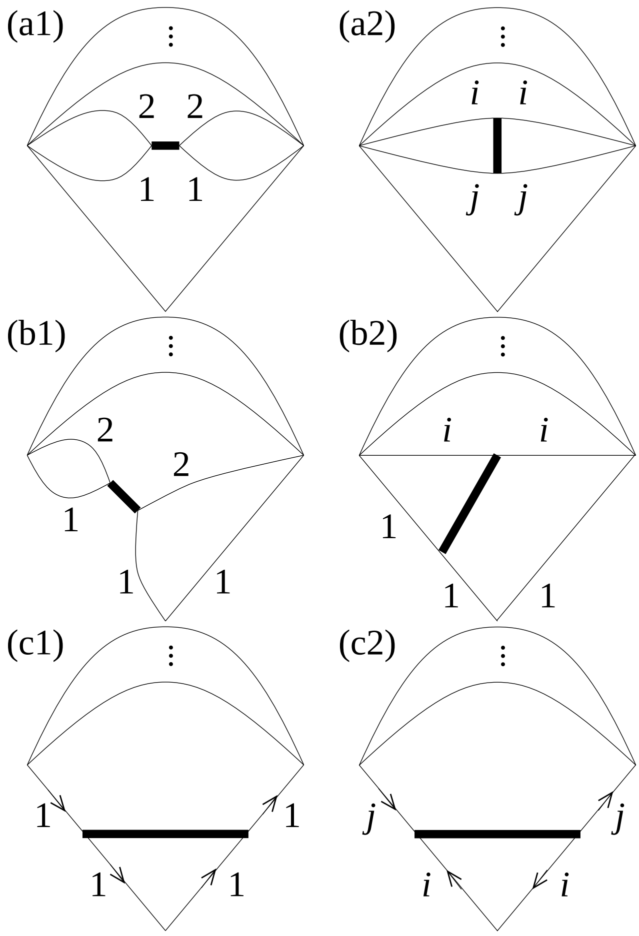


Figure 7: Order g^2 corrections to $\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)(\mathrm{tr}\,\bar{z}tz)(w)\rangle$ with $t_{22}=0$: (a) within the rainbow ($i,j=1,2$ in the second diagram); (b) from the rainbow to X^2 (there are similar ones with the other leg of X^2 uncorrected).

BPS) scalar operators of same scaling dimension Δ; then

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle = \frac{C}{(x-y)^{2\Delta}} \tag{8.9}$$

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\,\mathrm{tr}\,X^2(w)\rangle = \frac{\tilde{C}}{(x-y)^{2\Delta}}\frac{G(x,w)G(y,w)}{G(x,y)} \tag{8.10}$$

in $d=4$; $\mathrm{tr}\,X^2 = \mathrm{tr}\,\bar{z}tz$ as in equation (8.7). In other words, coordinate dependence (modulo the ratio of free scalar correlators) is the same, and the difference is just a constant factor. Assume now that $\mathcal{O}$ is constructed of only z_i-s (then $\bar{\mathcal{O}}$ is made of only $\bar{z}_j$-s). Then, the only $\mathcal{O}(g^2)$ contributions to the two-point function $\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle$ are proportional to $\tilde{B}(x,y)$. Similarly, the $\mathcal{O}(g^2)$ corrections to the correlator $\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\,\mathrm{tr}\,X^2(w)\rangle$ are proportional to $\tilde{B}(x,y)$ and to $[\tilde{C}(x;y,w)+\tilde{C}(y;x,w)]$. Index structure of these building blocks is the same (as discussed after equation (8.7), so

$$\frac{\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle|_{g^2}}{\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle|_{\mathrm{free}}} = \zeta\,\tilde{B}(x,y) \tag{8.11}$$

$$\frac{\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\,\mathrm{tr}\,X^2(w)\rangle|_{g^2}}{\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\,\mathrm{tr}\,X^2(w)\rangle|_{\mathrm{free}}} = \frac{1}{2}\zeta'\left[\tilde{C}(x;w,y)+\tilde{C}(y;w,x)+\tilde{B}(x,y)\right]. \tag{8.12}$$

The difference between $\tilde{B}(x,y)$ contributions to $\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\,\mathrm{tr}\,X^2(w)\rangle$ and to $\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle$ is that the pair of indices which go to w rather than stretch between x and y directly, is distinguished. Hence, $\zeta'=\zeta$ times some combinatorial factor.[12] As was discussed in section 6, $\tilde{B}$ and $\tilde{C}$ have the form $\tilde{B}(x,y)=\tilde{a}\log\frac{(x-y)^2}{\epsilon^2}+\tilde{b}$, $\tilde{C}(x;y,w)=\tilde{a}'\log\frac{(x-y)^2(x-w)^2}{\epsilon^4}-\tilde{a}''\log\frac{(y-w)^2}{\epsilon^2}+\tilde{b}'$. By comparing (8.10) and (8.9), we see that expression (8.12) must have the same coordinate dependence as (8.11). This restricts $\tilde{a}'=\tilde{a}''=\frac{1}{2}\tilde{a}$, which reproduces the "winking cat" identity (8.8).

Finally, we can relate $\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\,\mathrm{tr}\,X^2(w)\rangle$ to $\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle$ by a Ward identity. As shown in [24], the ratio

$$\frac{\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)T_{\mu\nu}(0)\rangle}{\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle} = \frac{2\Delta}{3\pi^2}\frac{t_{\mu\nu}(\gamma)(x-y)^4}{x^4y^4} \tag{8.13}$$

depends on the scaling dimension Δ of the operator $\mathcal{O}$ (here, $\gamma=\frac{x}{x^2}-\frac{y}{y^2}$ and $t_{\mu\nu}(\gamma)=\frac{\gamma_\mu\gamma_\nu}{\gamma^2}-\frac{1}{4}\eta_{\mu\nu}$). Since the energy momentum tensor $T_{\mu\nu}$ is in the same $\mathcal{N}=4$ multiplet with $\mathrm{tr}\,X^2$, there is also nothing peculiar about the fact that $\tilde{C}_{12}/C_{12}$ can in general receive $\mathcal{O}(g^2)$ correction. This ratio also depends on Δ.

8.1.2 General $\left\langle \mathcal{O}_{\frac{1}{4}}\mathcal{O}_{\frac{1}{4}}\mathcal{O}_{\frac{1}{2}}\right\rangle$ correlators

Three-point functions of two $\frac{1}{4}$-BPS operator and one $\frac{1}{2}$-BPS operator are similar to the ones described in section 8.1.1. It suffices to consider a single three-point function (such that the Clebsch-Gordon coefficient[13] for these three vectors in the given irreps of SU(4) is nonzero) for each set of three representations. Without loss of generality, we can choose a $[p,q,p]$ scalar composite $\mathcal{O}(x)$ to be made of only z-s; a $[r,s,r]$ scalar composite $\mathcal{O}'(y)$ to be made of only $\bar{z}$-s; and a $[0,k,0]$ scalar composite $\mathrm{tr}\,X^{\alpha_1+\alpha_2}$ at w of the form

$$t_{i_1\ldots i_{\alpha_1};j_1\ldots j_{\alpha_2}}\mathrm{str}\,z_{i_1}\ldots z_{i_{\alpha_1}}\bar{z}_{j_1}\ldots\bar{z}_{j_{\alpha_2}}, \tag{8.14}$$

where $\alpha_1+\alpha_2=k$, and $t_{i_1\ldots i_{\alpha_1};j_1\ldots j_{\alpha_2}}$ is the appropriate irreducible SU(3) tensor (like in [6]). The correlators we are after are

$$\langle \mathcal{O}(x)\mathcal{O}'(y)(\mathrm{tr}\,X^{\alpha_1+\alpha_2})(w)\rangle. \tag{8.15}$$

Position dependence of (8.15) is

$$[G(x,y)^{(2p+r)+(2r+s)-k}G(x,w)^{k+(2p+r)-(2r+s)}G(w,y)^{k-(2p+r)+(2r+s)}]^{1/2} \tag{8.16}$$

at Born level. The contributing free diagrams are similar to the ones shown in figure 6; and $\mathcal{O}(g^2)$ diagrams, to those of figure 7, but now there can be a different number of lines

[12] In particular, if $\Delta=\Delta_0$ is not corrected, then neither $\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\,\mathrm{tr}\,X^2(w)\rangle$ nor $\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle$ get any one loop corrections.

[13] By Wigner-Eckart theorem, for any three representations we only need to calculate one (nonvanishing) correlator of any representatives from these irreps.

stretching between x and w and between w and y. Apart from the factor (8.16), the general $\langle \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{2}} \rangle$ correlator (8.15) is given by

$$\alpha_{\text{free}} + \tilde{\beta}_{xy} \tilde{B}(x,y) + \tilde{\gamma}_x \tilde{C}(x;y,w) + \tilde{\gamma}_y \tilde{C}(y;x,w) + \mathcal{O}(g^4) \,. \tag{8.17}$$

According to the discussion of section 7, the remaining combinatorial coefficients vanish, $\beta_{xw} = \beta_{yw} = \gamma_w = 0$. Moreover, $\tilde{\gamma}_x = \tilde{\gamma}_y = -\tilde{\beta}_{xy}$ as follows from equations (7.4)–(7.6), so (8.17) reads

$$\alpha_{\text{free}} - \tilde{\beta}_{xy} \left(\tilde{C}(x;y,w) + \tilde{C}(y;x,w) - \tilde{B}(x,y) \right) + \mathcal{O}(g^4) \,. \tag{8.18}$$

Hence, we only need to verify that $\tilde{\beta}_{xy} = 0$.[14]

The simplifications we can use to deduce that $\tilde{\beta}_{xy} = 0$ without doing calculations, are discussed in section 7.2. Extremal three-point functions are always easy to identify, and with the BPS primaries in representations $[p,q,p]$, $[r,s,r]$, and $[0,k,0]$, the restrictions on the scaling dimension translate into

$$2r + s = 2p + q + k \,, \quad 2p + q = 2r + s + k \,, \quad \text{or} \quad 2p + q + 2r + s = k \,, \tag{8.19}$$

depending on which scaling dimension is the sum of the other two.

The "three flavor partition" boils down to being able to choose a single flavor (at Born level) for the lines between the two $\frac{1}{4}$-BPS operators, when the third operator is $\frac{1}{2}$-BPS. This is possible whenever

$$2r + s \leq k + q \quad \text{and} \quad 2p + q \leq k + s \,. \tag{8.20}$$

Alternatively, suppose we can choose the $\frac{1}{2}$-BPS operator $\mathcal{O}_w$ to be made of only $\bar{z}_1$-s and z_2-s; and the $\frac{1}{4}$-BPS operators as $\mathcal{O}_x$ of z_1-s and z_2-s, $\mathcal{O}_y$ of $\bar{z}_1$-s and $\bar{z}_2$-s. This is the "two flavor partition" of section 7.2. Flavors can be chosen this way if

$$k \leq q + s \,. \tag{8.21}$$

In all three cases (8.19), (8.20), and (8.21) there are no $\mathcal{O}(g^2)$ corrections, as established in section 7.2 using only SU(4) group theory and conformal invariance arguments. However, there are allowed three-point functions of the form $\langle \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{2}} \rangle$ where we can not choose irrep representatives in such a nice way.

Throughout the rest of this section, we will concentrate on the $\frac{1}{4}$-BPS operators with dimensions 7 and smaller, constructed in [1]. In particular, we will consider scalar composite operators in SU(4) representations of the form $[p,q,p]$, with $2p + q \leq 7$. These are [2,0,2], [2,1,2], [2,2,2], [3,1,3], and [2,3,2]. We will take $\frac{1}{2}$-BPS (single trace) operators as whichever ones are allowed by group theory. Of the triple products of the form $[p,q,p] \otimes [r,s,r] \otimes [0,k,0]$ containing the singlet, most satisfy at least one of the simplifying constraints (8.19), (8.20), or (8.21).[15] The exceptions are $[2,0,2] \otimes [2,0,2] \otimes [0,2,0]$ and $[3,1,3] \otimes [3,1,3] \otimes [0,4,0]$.

[14] The expression multiplying $\tilde{\beta}_{xy}$ in (8.18) is a nonzero, renormalization scale independent constant. In section 6, its value was computed to be $-\left(Y^2/32\pi^2\right)$.

[15] We omit the tedious details here. In order to find the allowed triple products, we used the method of BZ triangles, see appendix B. Then we just went through the list and checked if any of the conditions (8.19)–(8.21) applied.

Correlators of the form $\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)\,\mathrm{tr}\,X^2(w)\rangle$ were considered in section 8.1.1, so the only three-point function we actually have to calculate is $\langle \mathcal{O}_{[3,1,3]}\bar{\mathcal{O}}_{[3,1,3]}\,\mathrm{tr}\,X^{2+2}\rangle$. Explicitly, we can take

$$\mathcal{O}_x = \sum_{j=1}^{4} C_x^j \mathcal{O}_j\,, \qquad \mathcal{O}_y = \sum_{j=1}^{4} C_y^j \bar{\mathcal{O}}_j \quad \text{with} \quad \mathcal{O}_j \sim [z_1^4 z_2^2 z_3]\,, \tag{8.22}$$

$$\mathcal{O}_w \sim [z_2^2 \bar{z}_2^2] - \mathrm{SO}(6)\ \text{traces}\,, \tag{8.23}$$

to be the scalar composite operators[16] in the [3,1,3] of SU(4). With this choice of flavors, the free combinatorial factor for this three-point function is

$$\begin{aligned}
\alpha_{\text{free}} = \frac{(N^2-1)(N^2-2)}{41472N^2} \times \\
&\times(189540C_x^2C_y^2 - 4860C_x^2C_y^1N - 4860C_x^1C_y^2N - 131220C_x^4C_y^2N - 131220C_x^2C_y^4N + \\
&+ 360C_x^1C_y^1N^2 + 13500C_x^4C_y^1N^2 - 79380C_x^2C_y^2N^2 - 22680C_x^3C_y^2N^2 - \\
&- 22680C_x^2C_y^3N^2 + 5184C_x^3C_y^3N^2 + 13500C_x^1C_y^4N^2 - 30780C_x^4C_y^4N^2 + \\
&+ 2700C_x^2C_y^1N^3 - 270C_x^3C_y^1N^3 + 2700C_x^1C_y^2N^3 + 43740C_x^4C_y^2N^3 - \\
&- 270C_x^1C_y^3N^3 - 9720C_x^4C_y^3N^3 + 43740C_x^2C_y^4N^3 - 9720C_x^3C_y^4N^3 - \\
&- 115C_x^1C_y^1N^4 - 2760C_x^4C_y^1N^4 + 13500C_x^2C_y^2N^4 + 4410C_x^3C_y^2N^4 + \\
&+ 4410C_x^2C_y^3N^4 - 1332C_x^3C_y^3N^4 - 2760C_x^1C_y^4N^4 + 13680C_x^4C_y^4N^4 - \\
&- 450C_x^2C_y^1N^5 + 240C_x^3C_y^1N^5 - 450C_x^1C_y^2N^5 - 4500C_x^4C_y^2N^5 + \\
&+ 240C_x^1C_y^3N^5 + 2340C_x^4C_y^3N^5 - 4500C_x^2C_y^4N^5 + 2340C_x^3C_y^4N^5 - \\
&- 15C_x^1C_y^1N^6 - 990C_x^2C_y^2N^6 - 126C_x^3C_y^3N^6 - 1980C_x^4C_y^4N^6)
\end{aligned} \tag{8.24}$$

so $\langle \mathcal{O}_x\mathcal{O}_y\mathcal{O}_w\rangle \neq 0$ in general (and when $\mathcal{O}_x$ and $\mathcal{O}_y$ are $\frac{1}{4}$-BPS, in particular). We have also explicitly calculated[17] the $\mathcal{O}(g^2)$ combinatorial factor in (8.18):

$$\begin{aligned}
\tilde{\beta}_{xy} = \frac{(N^2-1)(N^2-4)}{13824} \times \\
&\times(-10800C_x^2C_y^1 + 4320C_x^3C_y^1 - 10800C_x^1C_y^2 - 259200C_x^4C_y^2 + 4320C_x^1C_y^3 + \\
&+ 103680C_x^4C_y^3 - 259200C_x^2C_y^4 + 103680C_x^3C_y^4 - 2025C_x^1C_y^1N - \\
&- 27000C_x^4C_y^1N - 32400C_x^2C_y^2N + 38880C_x^3C_y^2N + 38880C_x^2C_y^3N - \\
&- 25920C_x^3C_y^3N - 27000C_x^1C_y^4N - 129600C_x^4C_y^4N - 600C_x^2C_y^1N^2 + \\
&+ 2940C_x^3C_y^1N^2 - 600C_x^1C_y^2N^2 + 50400C_x^4C_y^2N^2 + 2940C_x^1C_y^3N^2 - \\
&- 7200C_x^4C_y^3N^2 + 50400C_x^2C_y^4N^2 - 7200C_x^3C_y^4N^2 + 175C_x^1C_y^1N^3 + \\
&+ 5400C_x^4C_y^1N^3 + 7200C_x^2C_y^2N^3 - 7920C_x^3C_y^2N^3 - 7920C_x^2C_y^3N^3 + \\
&+ 3888C_x^3C_y^3N^3 + 5400C_x^1C_y^4N^3 + 28800C_x^4C_y^4N^3 + 600C_x^2C_y^1N^4 - \\
&- 780C_x^3C_y^1N^4 + 600C_x^1C_y^2N^4 - 780C_x^1C_y^3N^4 - 2880C_x^4C_y^3N^4 - \\
&- 2880C_x^3C_y^4N^4 + 50C_x^1C_y^1N^5 + 288C_x^3C_y^3N^5)\,.
\end{aligned} \tag{8.25}$$

[16] $\mathcal{O}_{1,\ldots,4}$ were studied in [1], and the results are summarized here in appendix A.

[17] This calculation was done using *Mathematica* and took about 200 hours. The choice of flavors (8.22)–(8.23) was optimal from the computational efficiency point of view.

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If we choose the coefficients $(C_x^1, C_x^2, C_x^3, C_x^4)$ and $(C_y^1, C_y^2, C_y^3, C_y^4)$ independently from the set $\{(-\frac{12N}{N^2-2}, 1, -\frac{5}{N^2-2}, 0), (\frac{96}{N^2-4}, -\frac{4N}{N^2-4}, \frac{10N}{N^2-4}, 1)\}$, we recover $\tilde{\beta}_{xy} = 0$. This corresponds to taking $\mathcal{O}_x$ and $\mathcal{O}_y$ as the $\frac{1}{4}$-BPS chiral primaries found in [1], so there are no $\mathcal{O}(g^2)$ corrections in this case either.

8.2 Three-point functions of $\frac{1}{4}$-BPS operators

When all three operators are $\frac{1}{4}$-BPS, the arguments get progressively more tedious. We will chose $2l + k \leq 2p + q \leq 2r + s$. The simplifications discussed in section 7.2 applicable to correlators $\langle \mathcal{O}_{[p,q,p]} \mathcal{O}_{[r,s,r]} \mathcal{O}_{[l,k,l]} \rangle$ are: the extremality condition

$$2r + s = 2p + q + 2l + k\,; \tag{8.26}$$

and the "partition into three disjoint flavors" condition,[18]

$$\begin{cases} 2r + s & \leq & 2l + k + q \\ 2r + s & \leq & 2p + q + k \\ 2p + q & \leq & 2l + k + s \end{cases} \tag{8.27}$$

(the three inequalities have to be satisfied simultaneously). For example, (8.27) are true when all the $\frac{1}{4}$-BPS operators are in the $84 = [2, 0, 2]$ of SU(4); we can take[19]

$$\mathcal{Y}(x) = \left\{(\mathrm{tr}\, z_1^2)(\mathrm{tr}\, z_2^2) - (\mathrm{tr}\, z_1 z_2)(\mathrm{tr}\, z_1 z_2)\right\} + \frac{1}{N}\left\{\mathrm{tr}\, [z_1, z_2]^2\right\}, \tag{8.28}$$

$$\mathcal{Y}(y) = \left\{(\mathrm{tr}\, \bar{z}_1^2)(\mathrm{tr}\, \bar{z}_3^2) - (\mathrm{tr}\, \bar{z}_1 \bar{z}_3)(\mathrm{tr}\, \bar{z}_1 \bar{z}_3)\right\} + \frac{1}{N}\left\{\mathrm{tr}\, [\bar{z}_1, \bar{z}_3]^2\right\}, \tag{8.29}$$

$$\mathcal{Y}(w) = \left\{(\mathrm{tr}\, z_3^2)(\mathrm{tr}\, \bar{z}_2^2) - (\mathrm{tr}\, z_3 \bar{z}_2)(\mathrm{tr}\, z_3 \bar{z}_2)\right\} + \frac{1}{N}\left\{\mathrm{tr}\, [z_3, \bar{z}_2]^2\right\}. \tag{8.30}$$

The Born amplitude

$$\langle \mathcal{Y}(x)\ \mathcal{Y}(y)\ \mathcal{Y}(w) \rangle_{\mathrm{free}} \propto (N^2 - 1)(N^2 - 4)(2N^2 - 15) \tag{8.31}$$

does not vanish[20] for $N > 2$, so we can't blame the lack of corrections on group theory, and since the correlator $\langle \mathcal{Y}(x)\ \mathcal{Y}(y)\ \mathcal{Y}(w) \rangle$ is of the form (7.7), it receives no radiative corrections at order g^2.

Of the allowed $\langle \mathcal{O}_{\frac{1}{4}}(x) \mathcal{O}_{\frac{1}{4}}(y) \mathcal{O}_{\frac{1}{4}}(w) \rangle$ three-point functions where each $\mathcal{O}_{\frac{1}{4}}$ is a scalar composite in a $[p, q, p]$ of SU(4) with $2p + q \leq 7$, ten more satisfy (8.26) or (8.27).[21] For the remaining five correlators

$$\langle \mathcal{Y}_{[2,0,2]}(x) \mathcal{Y}_{[2,0,2]}(y) \mathcal{Y}_{[2,2,2]}(w) \rangle$$

[18] Which just says that the number of scalars exchanged between each pair of $\mathcal{O}$-s is no larger than the length of the first column in the corresponding Young tableaux. There are three more inequalities, but they are satisfied trivially since we took $2l + k \leq 2p + q \leq 2r + s$.

[19] As shown in appendix C, such operators are in fact in the 84 of SU(4).

[20] $N > 1$ in general since the gauge group is SU(N), while for $N = 2$ the single and double trace operators are proportional, and there are no $\frac{1}{4}$-BPS operators in the 84 of SU(4).

[21] We used the method of BZ triangles (see appendix B) to find the allowed triple products.

$$
\begin{aligned}
&\langle \mathcal{Y}_{[2,0,2]}(x)\mathcal{Y}_{[2,1,2]}(y)\mathcal{Y}_{[2,3,2]}(w)\rangle \\
&\langle \mathcal{Y}_{[2,0,2]}(x)\mathcal{Y}_{[2,1,2]}(y)\mathcal{Y}_{[3,1,3]}(w)\rangle \\
&\langle \mathcal{Y}_{[2,0,2]}(x)\mathcal{Y}_{[3,1,3]}(y)\mathcal{Y}_{[2,3,2]}(w)\rangle \\
&\langle \mathcal{Y}_{[2,0,2]}(x)\mathcal{Y}_{[3,1,3]}(y)\mathcal{Y}_{[3,1,3]}(w)\rangle
\end{aligned}
\tag{8.32}
$$

we have to verify that there are no contributions proportional to any of the functions $\tilde{B}(x,y)$, $\tilde{B}(x,w)$, or $\tilde{B}(y,w)$. In fact, with $2l+k \leq 2p+q \leq 2r+s$,

$$
2l+k \leq 2r+s+q \quad \text{and} \quad 2p+q \leq 2r+s+k \tag{8.33}
$$

are automatically satisfied, so we can always choose the operators as[22]

$$
\begin{aligned}
\mathcal{Y}_{[l,k,l]}(x) &\sim [\bar{z}_1^a \bar{z}_2^b z_3^e] && (8.34)\\
\mathcal{Y}_{[p,q,p]}(y) &\sim [\bar{z}_1^c \bar{z}_2^d \bar{z}_3^e] && (8.35)\\
\mathcal{Y}_{[r,s,r]}(w) &\sim [z_1^{r+s} z_2^r]\,, && (8.36)
\end{aligned}
$$

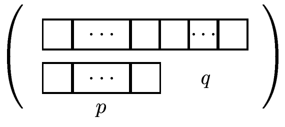
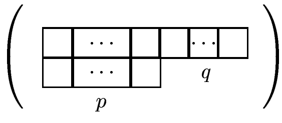
where $e \equiv \frac{1}{2}[(2l+k)+(2p+q)-(2r+s)] \leq l+k, p+q$; and integers a, b, c, d partition $r+s = a+c$, $r = b+d$. Then $\tilde{\beta}_{xy} = 0$ since the operators exchanged between $\mathcal{Y}_x$ and $\mathcal{Y}_y$ all have the same flavor, and we only need to calculate $\tilde{\beta}_{xw}$ and $\tilde{\beta}_{yw}$. Details of these calculations are given in appendix D, and here we just quote the result: as in the cases considered so far, $\tilde{\beta}_{xy} = \tilde{\beta}_{xw} = \tilde{\beta}_{yw} = 0$, and none of the three-point functions (8.32) receive any $\mathcal{O}(g^2)$ corrections.

9. $\left\langle \mathcal{O}_{\frac{1}{4}}\mathcal{O}_{\frac{1}{4}}\mathcal{O}_{\frac{1}{2}} \right\rangle$ correlators in the large-N limit

Like the two-point functions studied in [1], $\langle \mathcal{O}_x\mathcal{O}_y\mathcal{O}_w\rangle$ calculations get progressively more cumbersome as the representations of the $\mathcal{O}$-s become larger. In this section we will calculate correlators of two $\frac{1}{4}$-BPS operators with one $\frac{1}{2}$-BPS operator, in the large-N limit. The situation when all three operators are $\frac{1}{4}$-BPS is even less tractable, and we avoid it here.

9.1 Large-N operators

We will use the $\frac{1}{4}$-BPS operators found in [1]. Schematically, the special double and single trace operators can be written as

$\mathcal{O}_{[p,q,p]} \sim$  , $\mathcal{K}_{[p,q,p]} \sim$  (9.1)

(each continuous group of boxes stands for an SU(N) trace); explicit formulae for highest SU(4) weight operators of this form are given in appendix A.6. In the large-N limit the

[22] This choice of flavors was motivated by computational efficiency; our *Mathematica* calculations took more computer time with other flavor breakdowns.

linear combinations

$$\tilde{\mathcal{Y}}_{[p,q,p]} = \mathcal{K}_{[p,q,p]} + \mathcal{O}(N^{-2}) \tag{9.2}$$

$$\mathcal{Y}_{[p,q,p]} = \mathcal{O}_{[p,q,p]} - \frac{p(p+q)}{N}\mathcal{K}_{[p,q,p]} + \mathcal{O}(N^{-2}) \tag{9.3}$$

are eigenstates of the dilatations operator. $\mathcal{Y}_{[p,q,p]}$ have protected normalization and scaling dimension ($\Delta_{\mathcal{Y}} = 2p+q$) at order g^2, and were argued to be $\frac{1}{4}$-BPS.

We did not specify the SU(4) weights of operators $\mathcal{O}_{[p,q,p]}$ and $\mathcal{K}_{[p,q,p]}$ in (9.1). The choice of weights will depend on the representations in the triple product $[p,q,p]\otimes[r,s,r]\otimes[0,k,0]$ in the following way. Assume $p \leq r$; then it is convenient to choose[23]

$$\mathcal{Y}_{[p,q,p]}(x) \sim [\bar{z}_1^l z_2^n z_3^p] \tag{9.4}$$

$$\mathcal{Y}_{[r,s,r]}(y) \sim [\bar{z}_1^m \bar{z}_2^n \bar{z}_3^p] \tag{9.5}$$

$$\mathcal{O}_{[0,k,0]}(w) \sim [z_1^k] \tag{9.6}$$

with $l \equiv \frac{1}{2}[(2p+q)+k-(2r+s)]$, $m \equiv k-l$ and $n \equiv p+q-l$. We will also assume that none of the simplifications (8.19)–(8.21) apply, since those cases were already discussed in section 8.1.2.

9.2 $\left\langle \mathcal{K}\mathcal{K}\mathcal{O}_{\frac{1}{2}} \right\rangle_{\text{free}}$, $\left\langle \mathcal{K}\mathcal{O}\mathcal{O}_{\frac{1}{2}} \right\rangle_{\text{free}}$, $\left\langle \mathcal{O}\mathcal{K}\mathcal{O}_{\frac{1}{2}} \right\rangle_{\text{free}}$, and $\left\langle \mathcal{O}\mathcal{O}\mathcal{O}_{\frac{1}{2}} \right\rangle_{\text{free}}$

We can estimate the leading large N behavior of the combinatorial factors α_{free} and $\tilde{\beta}_{xy}$ using the "trace merging formula"

$$2\,(\operatorname{tr} At^c)\,(\operatorname{tr} t^c B) = \operatorname{tr} AB - \frac{1}{N}\,(\operatorname{tr} A)\,(\operatorname{tr} B)\,, \tag{9.7}$$

where A and B are arbitrary $N \times N$ matrices and t^c are SU(N) generators in the fundamental (sums on repeated indices are implied). With (9.7) and the expression for the quadratic Casimir[24]

$$T_r^c T_r^c = C_2(r)\,\mathbf{1} \tag{9.8}$$

of SU(N), we find for example that for $k \geq 2$,[25]

$$\begin{aligned}(\operatorname{tr} t^{a_1} t^{a_2} \ldots t^{a_k})(\operatorname{tr} t^{a_k} \ldots t^{a_2} t^{a_1}) &= \left(\frac{N^2-1}{2N}\right)^k - \left(\frac{N^2-1}{2N}\right)\left(\frac{-1}{2N}\right)^{k-1} \\ &= \left(\frac{N}{2}\right)^k \left[1 + \mathcal{O}(1/N^2)\right].\end{aligned} \tag{9.9}$$

To have this large-N behavior, generators in the two traces should appear in opposite order. When the generators are taken in any other order (except cyclic permutations inside the traces), such products are suppressed by at least a power of N^2.

[23] With this choice of flavors, the manipulations of the following sections are a simple generalization of the arguments of section 9 of [1].

[24] In particular, the adjoint and fundamental representations will be of interested, and for these $C_2(\text{adj}) = N$, $C_2(\text{fund}) = (\frac{N^2-1}{2N})$, $C(\text{adj}) = N$, $C(\text{fund}) = \frac{1}{2}$.

[25] One can derive (9.9) from a one-term recursion relation defined by (9.7), (9.8), and $\operatorname{tr} t^a = 0$.

Calculations proceed along the same lines as in [1]. We begin by considering correlators of the form $\langle \mathcal{O}\mathcal{O}\mathcal{O}_{\frac{1}{2}} \rangle$ with the two $\mathcal{O}$-s in the same representation $[p,q,p]$. In this case the leading contribution to α_{free} comes from terms like

$$\begin{aligned}(\operatorname{tr} t^{a_1}\dots t^{a_l}t^{b_1}\dots t^{b_n})(\operatorname{tr} t^{c_1}\dots t^{c_p})(\operatorname{tr} t^{d_1}\dots t^{d_l}t^{b_n}\dots t^{b_1})\times \\ \times(\operatorname{tr} t^{c_p}\dots t^{c_1})(\operatorname{tr} t^{a_l}\dots t^{a_1}t^{d_l}\dots t^{d_1}) \sim \\ \sim \left(\frac{1}{2}\right)^2 N \left(\frac{N}{2}\right)^{2l-2+n} \times \left(\frac{N}{2}\right)^{p} = \left(\frac{1}{2}\right)\left(\frac{N}{2}\right)^{2l+n+p-1} .\end{aligned} \tag{9.10}$$

The factor of $(1/2)^2$ comes about because we merge traces twice; the exponent $2l+n-2$ counts how many generators collapse using $t^c t^c \sim (1/2)N\mathbf{1}$; the extra factor of N is due to $\operatorname{tr}\mathbf{1} = N$; and finally $(N/2)^p$ is from contracting the traces of equal length containing the t^{c_i}-s. All remaining calculations of this and next sections are analogous, and we won't spell things out as much.

If the representations of $[p,q,p]$ and $[r,s,r]$ are different, a similar situation occurs when for example $p = r+s$, i.e. $\mathcal{O}_{[p,q,p]}$ and $\mathcal{O}_{[r,s,r]}$ contain traces of equal length. Then we merge traces twice, and one set of traces collapses completely as in (9.9). Otherwise, we have to merge traces three times, so the leading contributions to $\langle \mathcal{O}\mathcal{O}\mathcal{O}_{\frac{1}{2}} \rangle_{\text{free}}$ are of the form

$$\begin{aligned}&(\operatorname{tr} t^{a_1}\dots t^{a_l}t^{b_1}\dots t^{b_n})(\operatorname{tr} t^{c_1}\dots t^{c_p})\ (\operatorname{tr} t^{d_1}\dots t^{d_m}t^{b_n}\dots)(\operatorname{tr}\dots t^{b_1}t^{c_p}\dots t^{c_1})\times \\ &\times(\operatorname{tr} t^{a_l}\dots t^{a_1}t^{d_m}\dots t^{d_1}) \sim \\ &\qquad \sim \left(\frac{1}{2}\right)\left(\frac{N}{2}\right)^{p+l+m+n-1} \quad \text{if } p=r \text{ or } p+q=r+s \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{or } p=r+s \text{ or } r=p+q \\ &\qquad \sim \left(\frac{1}{2}\right)^3\left(\frac{N}{2}\right)^{p+l+m+n-3} \quad \text{otherwise.}\end{aligned} \tag{9.11}$$

For the other three types of correlators, no pair of traces ever collapses completely, so the answers are more uniform. We find that the large-N behavior of $\langle \mathcal{K}\mathcal{K}\mathcal{O}_{\frac{1}{2}} \rangle_{\text{free}}$ is defined by terms like

$$\begin{aligned}&(\operatorname{tr} t^{a_1}\dots t^{a_l}t^{b_1}\dots t^{b_n}t^{c_1}\dots t^{c_p})\ (\operatorname{tr} t^{d_1}\dots t^{d_m}t^{b_n}\dots t^{b_1}t^{c_p}\dots t^{c_1})\ (\operatorname{tr} t^{a_l}\dots t^{a_1}t^{d_m}\dots t^{d_1}) \sim \\ &\sim \left(\frac{1}{2}\right)^2\left(\frac{N}{2}\right)^{p+l+m+n-2} N = \left(\frac{1}{2}\right)\left(\frac{N}{2}\right)^{p+l+m+n-1}\end{aligned} \tag{9.12}$$

as we merge traces twice. Similarly, $\langle \mathcal{O}\mathcal{K}\mathcal{O}_{\frac{1}{2}} \rangle_{\text{free}}$ scales as the terms

$$\begin{aligned}&(\operatorname{tr} t^{a_1}\dots t^{a_l}t^{b_1}\dots t^{b_n})(\operatorname{tr} t^{c_1}\dots t^{c_p})(\operatorname{tr} t^{d_1}\dots t^{d_m}t^{b_n}\dots t^{b_1}t^{c_p}\dots t^{c_1})\times \\ &\times(\operatorname{tr} t^{a_l}\dots t^{a_1}t^{d_m}\dots t^{d_1}) \sim \left(\frac{1}{2}\right)^2\left(\frac{N}{2}\right)^{p+l+m+n-2}\end{aligned} \tag{9.13}$$

since traces have to be merged three times now. The three-point functions $\langle \mathcal{K}\mathcal{O}\mathcal{O}_{\frac{1}{2}} \rangle_{\text{free}}$ also have the leading large-N dependence (9.13).

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9.3 $\left\langle \mathcal{KKO}_{\frac{1}{2}} \right\rangle_{g^2}$, $\left\langle \mathcal{KOO}_{\frac{1}{2}} \right\rangle_{g^2}$, $\left\langle \mathcal{OKO}_{\frac{1}{2}} \right\rangle_{g^2}$, and $\left\langle \mathcal{OOO}_{\frac{1}{2}} \right\rangle_{g^2}$

Here there are no special cases to consider. We have to merge traces twice for $\langle \mathcal{KKO}_{\frac{1}{2}} \rangle_{g^2}$, three times for $\langle \mathcal{KOO}_{\frac{1}{2}} \rangle_{g^2}$ or $\langle \mathcal{OKO}_{\frac{1}{2}} \rangle_{g^2}$, and four times for $\langle \mathcal{OOO}_{\frac{1}{2}} \rangle_{g^2}$. The leading behavior of the $\tilde{\beta}_{xy}$ combinatorial coefficient for the three-point functions $\langle \mathcal{KKO}_{\frac{1}{2}} \rangle_{g^2}$ is the same as for terms of the form

$$\begin{aligned}
&(\mathrm{tr}\, t^{a_1} \ldots t^{a_l} t^{b_1} \ldots t^{b_{n-1}} [t^{c_1}, t^s] t^{c_1} \ldots t^{c_p}) \ (\mathrm{tr}\, t^{d_1} \ldots t^{d_m} t^{c_p} \ldots t^{c_2} [t^{b_n}, t^s] t^{b_n} \ldots t^{b_1}) \times \\
&\times (\mathrm{tr}\, t^{a_l} \ldots t^{a_1} t^{d_m} \ldots t^{d_1}) \\
&\qquad \sim \left(\frac{1}{2}\right)^2 \left(\frac{N}{2}\right)^{l+m-2} (\mathrm{tr}\, t^{b_1} \ldots t^{b_{n-1}} [t^{c_1}, t^s] t^{c_1} \ldots t^{c_p} t^{c_p} \ldots t^{c_2} [t^{b_n}, t^s] t^{b_n} \ldots t^{b_1}) \\
&\qquad \sim \left(\frac{1}{2}\right)^2 \left(\frac{N}{2}\right)^{p+l+m+n-4} (\mathrm{tr}\, [t^{c_1}, t^s] t^{c_1} [t^{b_n}, t^s] t^{b_n}) \\
&\qquad \sim \left(\frac{1}{2}\right) \left(\frac{N}{2}\right)^{p+l+m+n}
\end{aligned} \tag{9.14}$$

which give the leading large-N contributions to it. In the same fashion, the most significant terms in the correlators $\langle \mathcal{OKO}_{\frac{1}{2}} \rangle_{g^2}$ are

$$\begin{aligned}
&(\mathrm{tr}\, t^{a_1} \ldots t^{a_l} t^{b_1} \ldots t^{b_{n-1}} [t^{c_1}, t^s])(\mathrm{tr}\, t^{c_1} \ldots t^{c_p})(\mathrm{tr}\, t^{d_1} \ldots t^{d_m} t^{c_p} \ldots t^{c_2} [t^{b_n}, t^s] t^{b_n} \ldots t^{b_1}) \times \\
&\times (\mathrm{tr}\, t^{a_l} \ldots t^{a_1} t^{d_m} \ldots t^{d_1}) \\
&\qquad \sim \left(\frac{1}{2}\right)^2 \left(\frac{N}{2}\right)^{p+l+m+n-1} \sim \langle \mathcal{KOO}_{\frac{1}{2}} \rangle_{g^2}
\end{aligned} \tag{9.15}$$

while $\langle \mathcal{OOO}_{\frac{1}{2}} \rangle_{g^2}$ gets its leading N behavior from terms like

$$\begin{aligned}
&(\mathrm{tr}\, t^{a_1} \ldots t^{a_l} t^{b_1} \ldots t^{b_{n-1}} [t^{c_1}, t^s])(\mathrm{tr}\, t^{c_1} \ldots t^{c_p})(\mathrm{tr}\, t^{d_1} \ldots t^{d_m} t^{c_p} \ldots) \times \\
&\times (\mathrm{tr}\, \ldots t^{c_2} [t^{b_n}, t^s] t^{b_n} \ldots t^{b_1})(\mathrm{tr}\, t^{a_l} \ldots t^{a_1} t^{d_m} \ldots t^{d_1}) \\
&\qquad \sim \left(\frac{1}{2}\right)^3 \left(\frac{N}{2}\right)^{p+l+m+n-2} .
\end{aligned} \tag{9.16}$$

9.4 $\left\langle \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{2}} \right\rangle$ correlators are protected

With just a little more work, we can find the ratios of the $\mathcal{O}(g^2)$ corrections to the three-point functions $\langle \mathcal{OOO}_{\frac{1}{2}} \rangle_{g^2}$, $\langle \mathcal{OKO}_{\frac{1}{2}} \rangle_{g^2}$, $\langle \mathcal{KOO}_{\frac{1}{2}} \rangle_{g^2}$, and $\langle \mathcal{KKO}_{\frac{1}{2}} \rangle_{g^2}$. The argument proceeds along the same lines as in [1]. Given a term with generators in a particular order, contributing to $\langle \mathcal{KKO}_{\frac{1}{2}} \rangle_{g^2}$, such as the one shown in (9.14), we know that a term with the same order of generators also gives a leading contribution to $\langle \mathcal{OKO}_{\frac{1}{2}} \rangle_{g^2}$ as in (9.15). However, cyclic permutations within the two traces (of length p and $p+q$) of $\mathcal{O}$, also contribute to $\langle \mathcal{OKO}_{\frac{1}{2}} \rangle_{g^2}$ in the same amount as the term (9.15). Therefore,

$$\langle \mathcal{OKO}_{\frac{1}{2}} \rangle_{g^2} / \langle \mathcal{KKO}_{\frac{1}{2}} \rangle_{g^2} = \frac{p(p+q)}{N} + \mathcal{O}(N^{-3}) \equiv \beta\,. \tag{9.17}$$

In the same fashion

$$\langle \mathcal{K}\mathcal{O}\mathcal{O}_{\frac{1}{2}}\rangle_{g^2}/\langle \mathcal{K}\mathcal{K}\mathcal{O}_{\frac{1}{2}}\rangle_{g^2} = \frac{r(r+s)}{N} + \mathcal{O}(N^{-3}) \equiv \beta' \tag{9.18}$$

and

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}_{\frac{1}{2}}\rangle_{g^2}/\langle \mathcal{K}\mathcal{O}\mathcal{O}_{\frac{1}{2}}\rangle_{g^2} = \frac{p(p+q)}{N} + \mathcal{O}(N^{-3})\,. \tag{9.19}$$

Next consider the Born level correlators of section 9.2. When a pair of traces collapses completely (see equations 9.11)–(9.13)), we get

$$\langle \mathcal{Y}_{[p,q,p]}\mathcal{Y}_{[r,s,r]}\mathcal{O}_{\frac{1}{2}}\rangle_{\text{free}} \sim \langle \mathcal{O}\mathcal{O}\mathcal{O}_{\frac{1}{2}}\rangle_{\text{free}} \sim N^{p+l+m+n-1}\,. \tag{9.20}$$

Otherwise, the contributions add up to

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}_{\frac{1}{2}}\rangle_{\text{free}} - \beta\langle \mathcal{K}\mathcal{O}\mathcal{O}_{\frac{1}{2}}\rangle_{\text{free}} - \beta'\langle \mathcal{O}\mathcal{K}\mathcal{O}_{\frac{1}{2}}\rangle_{\text{free}} + \beta\beta'\langle \mathcal{K}\mathcal{K}\mathcal{O}_{\frac{1}{2}}\rangle_{\text{free}} \sim N^{p+l+m+n-3}\,. \tag{9.21}$$

The terms in (9.21) are all of the same order and do not cancel. The factors of β and β' discussed above are still present, but there are other complications. First, the string of t^c-s can be inserted anywhere in the third trace in (9.11), and cyclic permutations of the t^b-s in the same trace give terms of the same order in N. This results in an extra factor of $(r-p)^2$. Second, different terms in the sum over antisymmetrizations (as in equation (A.29), for example) contribute differently. The combinatorics is more involved, and we do not discuss this case in detail.

Bringing everything together, we see that the order g^2 corrections to the three-point function of the BPS operators in the large-N limit add up to

$$\begin{aligned}\langle \mathcal{Y}_{[p,q,p]}\mathcal{Y}_{[r,s,r]}\mathcal{O}_{\frac{1}{2}}\rangle_{g^2} &\propto \left(\frac{1}{2}\right)\left(\frac{N}{2}\right)^{p+l+m+n-1}\tilde{B}(x,y)N\,\times \\ &\quad\times \begin{pmatrix}-\beta \\ 1\end{pmatrix}^t \begin{pmatrix}1+\mathcal{O}(N^{-2}) & \beta' \\ \beta & \beta\beta'\end{pmatrix}\begin{pmatrix}-\beta' \\ 1\end{pmatrix} \\ &= \left(\frac{1}{2}\right)\left(\frac{N}{2}\right)^{p+l+m+n-1}\tilde{B}(x,y)N\times\mathcal{O}(N^{-4})\,.\end{aligned} \tag{9.22}$$

A comparison of (9.22) with (9.20) or (9.21) shows that order g^2 corrections to three-point functions of one $\frac{1}{2}$-BPS operator with two $\frac{1}{4}$-BPS operators vanish in the large-N limit, within working precision.

10. Supergravity considerations

In the AdS/CFT correspondence, there is a duality mapping single trace $\frac{1}{2}$-BPS primary operators $\operatorname{tr} X^k$ of the SYM theory onto canonical supergravity fields, [3, 4]. Given a set of such $\frac{1}{2}$-BPS primary operators, one can compute their two- and three-point functions in SYM. Two-point functions define the normalization of operators, and three-point functions probe the interactions between them. Independently, both the normalization of the SUGRA fields as well as their couplings, can be read off from the supergravity action (or supergravity equations of motion), [5]. So as a check of the AdS/CFT correspondence, one can compare the unambiguously defined three- and higher n-point functions of normalized $\frac{1}{2}$-BPS operators in SYM, with the correlators of the corresponding elementary excitations in supergravity, [5, 14, 26, 27].

We would like to proceed, in the same spirit, with the $\frac{1}{4}$-BPS chiral primaries of the $\mathcal{N}=4$ Super Yang Mills calculated in this paper and in [1]. We argued that these two- and three-point functions are independent of the SYM coupling constant (at least to order g^2), so it is reasonable to expect these correlators to agree with their dual AdS description. However, multiple trace operators do not correspond to any of the fields appearing in the supergravity action, so the discussion will be different than in the case of the previously studied $\frac{1}{2}$-BPS primary operators $\mathrm{tr}\, X^k$.

10.1 OPE definition of $\frac{1}{4}$-BPS chiral primaries

One of the ways to see $\frac{1}{4}$-BPS chiral primaries is to consider higher n-point correlators of $\frac{1}{2}$-BPS operators. For example, four-point functions of [0,2,0] operators reveal a pole corresponding to the exchange of a [2,0,2] operator with a protected dimension $\Delta = 4$, [10]. In general, the $\frac{1}{4}$-BPS primaries $\mathcal{Y}_{[p,q,p]}$ show up in the four-point functions

$$\langle \mathrm{tr}\, X^{(p+q)}(x) \mathrm{tr}\, X^p(x+\epsilon)\ \mathrm{tr}\, X^{(p+q)}(y)\ \mathrm{tr}\, X^p(w) \rangle \tag{10.1}$$

in the limit $\epsilon \to 0$, as the $[p,q,p]$ operators with the threshold value of scaling dimension $\Delta = 2p+q = \dim[\mathrm{tr}\, X^{(p+q)}] + \dim[\mathrm{tr}\, X^p]$. In other words, $\frac{1}{4}$-BPS chiral primaries can be defined by the OPE-s of $\frac{1}{2}$-BPS operators as

$$\mathcal{P}^{\Delta=2p+q}_{[p,q,p]} \left[\lim_{\epsilon\to 0} \mathrm{tr}\, X^{(p+q)}(x)\ \mathrm{tr}\, X^p(x+\epsilon) \right] = \sum_i c_i \mathcal{Y}^{[p,q,p]}_i(x)\,. \tag{10.2}$$

Here, $\mathcal{P}^{\Delta}_{[p,q,p]}$ projects onto the $[p,q,p]$ representation of SU(4), and eliminates operators with scaling dimension other than Δ (e.g. the non-chiral descendants with the same SU(4) quantum numbers). Singular terms normally subtracted from an OPE such as (10.2), are automatically removed by applying $\mathcal{P}^{\Delta=2p+q}_{[p,q,p]}$.

On the other hand, one can see by calculating three-point correlators that all $\frac{1}{4}$-BPS primary operators $\mathcal{Y}^{[p,q,p]}_i$ are present in the OPE (10.2). It appears that for general N, there is no canonical definition of the special $\mathcal{Y}_{[p,q,p]}$ that is a linear combination of the single and double-trace scalar composite $[p,q,p]$ operators only. However, this $\mathcal{Y}_{[p,q,p]}$ dominates in the $N \to \infty$ limit. For large N, all other terms in the right hand side of (10.2) are suppressed by at least a factor of $1/N$, and the predominantly double-trace $\frac{1}{4}$-BPS chiral primary operator

$$\mathcal{Y}_{[p,q,p]} = \left(\begin{array}{l} \Box\cdots\Box\,\Box\,\Box\cdots\Box \\ \underbrace{\Box\cdots\Box}_{p} \qquad q \end{array} \right) + \mathcal{O}(1/N) \tag{10.3}$$

is uniquely defined by the OPE of $\frac{1}{2}$-BPS primaries.

When translated into the SUGRA language, the definition (10.2) implies that $\frac{1}{4}$-BPS primary operators of SYM should be thought of as threshold bound states of elementary SUGRA excitations. A threshold bound state is a state whose mass is precisely equal

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to the sum of the masses of all its constituents, and thus occurs at the lower end of the spectrum. Any bound state of BPS states which is itself BPS is automatically a threshold bound state. A familiar example is provided by an assembly of like sign charged Prasad-Sommerfield magnetic monopoles, whose classical static solution forms a threshold bound state of monopole constituents.

10.2 $\left\langle \mathcal{O}_{\frac{1}{4}}\mathcal{O}_{\frac{1}{4}} \right\rangle$ and $\left\langle \mathcal{O}_{\frac{1}{2}}\mathcal{O}'_{\frac{1}{2}}\mathcal{O}_{\frac{1}{4}} \right\rangle$ correlators

We are now going to illustrate the consistency of this dictionary. Specifically, we will look at two- and three-point functions involving $\frac{1}{2}$- and $\frac{1}{4}$-BPS operators in the large-N limit, that we calculated earlier in this paper and in [1], in $\mathcal{N}$=4 SYM. Then we will compare these correlators with their dual supergravity description.

The normalization of $\frac{1}{2}$- and $\frac{1}{4}$-BPS operators comes from their two-point functions, whose leading large-N behavior is [5, 1]

$$\langle \mathrm{tr}\, X^q(x)\ \mathrm{tr}\, X^q(y)\rangle \sim \frac{N^q}{(x-y)^{2q}} \tag{10.4}$$

$$\langle \mathcal{Y}_{[p,q,p]}(x)\ \bar{\mathcal{Y}}^{[p,q,p]}(y)\rangle \sim \frac{N^{(2p+q)}}{(x-y)^{2(2p+q)}} \tag{10.5}$$

times some N-independent factors which we omit.

The simplest three point functions involving $\frac{1}{2}$- and $\frac{1}{4}$-BPS operators are of the form $\langle \mathcal{O}_{\frac{1}{2}}(x)\mathcal{O}'_{\frac{1}{2}}(y)\mathcal{O}_{\frac{1}{4}}(w)\rangle$. If the SU($N$) traces collapse completely (in which case $\langle \mathcal{O}_{\frac{1}{2}}\mathcal{O}'_{\frac{1}{2}}\mathcal{O}_{\frac{1}{4}}\rangle$ are extremal), the normalized three point-functions are then

$$\frac{1}{\sqrt{N^{(2p+q)+(p+q)+p}}}\langle \mathrm{tr}\, X^{(p+q)}(x)\ \mathrm{tr}\, X^p(y)\ \mathcal{Y}_{[p,q,p]}(w)\rangle \sim 1 \tag{10.6}$$

from a field theory calculation; the space-time coordinate dependence is fixed by conformal invariance, so we will not exhibit it anymore. If the traces do not collapse completely, the correlator is suppressed by $1/N^2$ (see the discussion around equations (9.10)–(9.11) of section 9), and

$$\frac{1}{\sqrt{N^{(2p+q)+(k+l)+k}}}\langle \mathrm{tr}\, X^{(k+l)}(x)\ \mathrm{tr}\, X^k(y)\ \mathcal{Y}_{[p,q,p]}(w)\rangle \sim \frac{1}{N^2} \tag{10.7}$$

whenever $k \neq p$ of $l \neq q$. All this matches nicely with the corresponding supergravity diagrams in figure 8.

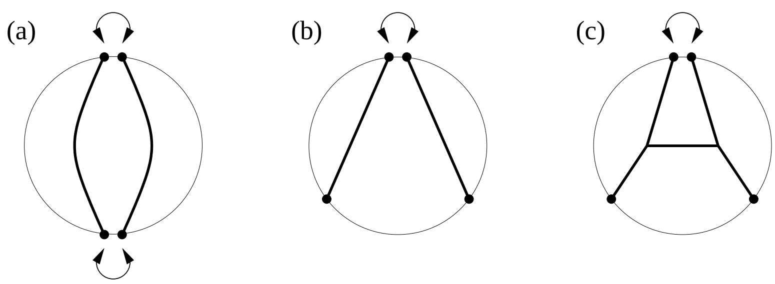

Figure 8: Leading AdS diagrams for (a) equation (10.5); (b) equation (10.6); (c) equation (10.7). Each cubic bulk interaction vertex goes like $1/N$.

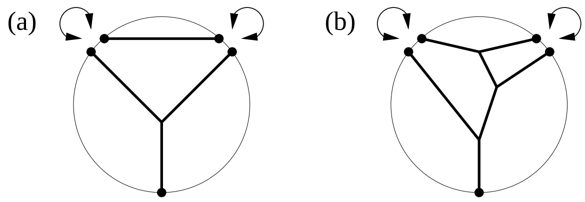

Figure 9: AdS description of equation (10.8).

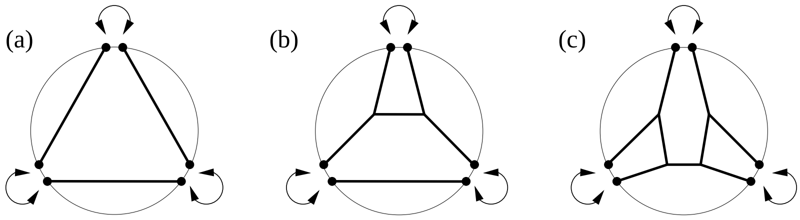

Figure 10: AdS description of equation (10.9).

We denoted the $\frac{1}{2}$-BPS operators by "$\bullet$"; and the predominantly double trace $\frac{1}{4}$-BPS primaries which arise from bringing two $\frac{1}{2}$-BPS operators together by " ".

There are also AdS diagrams with quartic interactions in the bulk, which have the same large-N dependence as in figure 8*c*; we will not show these.

10.3 $\left\langle \mathcal{O}_{\frac{1}{4}} \mathcal{O}'_{\frac{1}{4}} \mathcal{O}_{\frac{1}{2}} \right\rangle$ correlators

Other three point functions involving $\frac{1}{4}$-BPS as well as $\frac{1}{2}$-BPS operators can be analyzed similarly. Whenever traces of the SYM operators do not collapse completely, the supergravity counterparts of such correlators have extra bulk interaction vertices. The leading dependence of such correlators is then suppressed by the corresponding power of $1/N$. For example, correlators of the form $\langle \mathcal{O}_{\frac{1}{4}}(x) \mathcal{O}'_{\frac{1}{4}}(y) \mathcal{O}_{\frac{1}{2}}(w) \rangle$, discussed in section 9, behave like

$$\frac{1}{\sqrt{N^{(2p+q)+(2r+s)+k}}} \langle \mathcal{Y}_{[p,q,p]} \mathcal{Y}_{[r,s,r]} \mathrm{tr}\, X^k \rangle \sim \begin{cases} 1/N & (a) \text{ if one pair of traces} \\ & \quad \text{collapses completely} \\ 1/N^3 & (b) \text{ otherwise} \end{cases} \tag{10.8}$$

From the AdS point of view, this difference is explained by the diagrams in figure 9.

10.4 $\left\langle \mathcal{O}_{\frac{1}{4}} \mathcal{O}'_{\frac{1}{4}} \mathcal{O}''_{\frac{1}{4}} \right\rangle$ correlators

Similar arguments show that when all operators are $\frac{1}{4}$-BPS, the normalized three-point functions are

$$\frac{1}{\sqrt{N^{(2p+q)+(2r+s)+(2l+k)}}} \langle \mathcal{Y}_{[p,q,p]} \mathcal{Y}_{[r,s,r]} \mathcal{Y}_{[l,k,l]} \rangle \sim \begin{cases} 1 & (a) \text{ if all traces} \\ & \quad \text{collapse pairwise} \\ 1/N^2 & (b) \text{ if only one pair} \\ & \quad \text{of traces collapses} \\ 1/N^4 & (c) \text{ otherwise} \end{cases} \tag{10.9}$$

and the corresponding AdS diagrams in figure 10 show the correct leading large-N behavior.

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10.5 Detailed agreement between SYM and AdS

Unlike the $\frac{1}{2}$-BPS calculations (e.g. [5]), this study does not provide a new independent check or application of the AdS/CFT correspondence. On the one hand, the definition of the predominantly double-trace $\frac{1}{4}$-BPS operators in the SYM theory (in the large N limit) is based on the OPE of $\frac{1}{2}$-BPS primaries. On the other hand, AdS correlators of the duals of the $\frac{1}{4}$-BPS operators (bound states of elementary SUGRA excitations) are defined by the corresponding correlators of primary supergravity fields. Therefore, SYM correlators involving $\frac{1}{4}$-BPS operators agree by construction with their SUGRA counterparts.

This is especially clear in the cases show in figures 8a,b, 9a, and 10a. To leading order in N, these two- and three-point functions of $\frac{1}{4}$-BPS scalar composite operators are expressed in terms of the previously studied two- and three-point functions of $\frac{1}{2}$-BPS chiral primary operators.

11. Conjectures

Let us summarize what has been done so far. First, $\frac{1}{4}$-BPS primary operators were identified in [1]: for general representations $[p,q,p]$ in the large-N limit; and for general N in the case when $2p+q \leq 7$. Second, three-point functions involving $\frac{1}{2}$-BPS operators as well as several infinite families of $\frac{1}{4}$-BPS operators were considered in this paper, also for arbitrary N. It was found that there are no $\mathcal{O}(g^2)$ corrections to such correlators. Next, all three-point functions involving the $\frac{1}{4}$-BPS primaries with $2p+q \leq 7$ were computed for general N, and were shown to be protected at order g^2. In the large-N approximation, three point functions involving two $\frac{1}{4}$-BPS primaries and one $\frac{1}{2}$-BPS primary were shown to receive no $\mathcal{O}(g^2)$ corrections, for general representations of the operators involved. Finally, we presented AdS considerations which reproduced many features of the CFT two- and three-point functions in the large-N limit.

Collecting all the non-renormalization effects established above generates strong evidence for a number of natural conjectures, which we now state:

1. We conjecture that on the CFT side, for every $[p,q,p]$ representation of SU(4) and arbitrary N, there are $\frac{1}{4}$-BPS chiral primaries. Within each $[p,q,p]$, one of these operators is a linear combination of double and single trace scalar composites only; the other $\frac{1}{4}$-BPS chiral primaries in $[p,q,p]$ also involve operators with higher numbers of traces.

2. We speculate that two-point functions of $\frac{1}{4}$-BPS operators, as well as three-point functions involving $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS operators, do not depend on the coupling g^2 of $\mathcal{N}$=4 SYM. This non-renormalization also persist for all N, and is not just a large-N approximation.

3. One of the group theory arguments of section 7.2, and the analysis of section 9, generalize straightforwardly to extremal correlators, i.e. $(n+1)$-point functions of the form $\langle \mathcal{O}_0(x_0)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle$ with $\Delta_0 = \Delta_1 + \ldots + \Delta_n$. So do the AdS considerations of section 10. Therefore, we conjecture that arbitrary extremal correlators of $\frac{1}{2}$- and $\frac{1}{4}$-BPS chiral primaries are also protected.

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A. $\frac{1}{4}$-BPS operators explicitly

Scalar composite gauge invariant operators $\mathcal{O} \sim [(\phi^1)^{p+q}(\phi^2)^p]$ corresponding to highest weights of representations[26] $[p, q, p]$ of the R-symmetry group SU(4) $\sim$ SO(6) were studied in [1]. It was found that there are many such gauge invariant operators $\mathcal{O}_i$ for a given set of fields. They depend on how we partition the products into traces.

In general, none of the $\mathcal{O}_i$-s are eigenstates of the dilatations operator. To construct operators with a well defined scaling dimension, one has to take particular linear combinations, $\mathcal{Y}_j = \sum_i C^i_j \mathcal{O}_i$. Some of the $\mathcal{Y}_j$-s are $\frac{1}{4}$-BPS: they are annihilated by a quarter of the Poincaré supercharges, they are not descendants of $\frac{1}{8}$-BPS primaries, and at order g^2 they have a protected scaling dimension $\Delta = 2p + q$; their normalization also receives no $\mathcal{O}(g^2)$ corrections.

In this Appendix we list some of the results of [1]. In particular, we define the operators $\mathcal{O}_i$ for each representation $[p, q, p]$ with $2p + q \leq 7$, and list their linear combinations that are $\frac{1}{4}$-BPS. (We do not show the other pure operators here.) We reproduce the polynomial scalar composite operators of highest SU(4) weight found in [1], for every such $[p, q, p]$ representation. More precisely, shown below are their projections onto the SU(3) $\times$ U(1) subgroup of SU(4), which is the manifest symmetry group of the lagrangian (2.1). This is done for general N. We also write out the two special scalar composite operators which contribute to the $\frac{1}{4}$-BPS primaries in the large-N limit, for all $[p, q, p]$ representations.

A.1 [2, 0, 2]

There are two linearly independent gauge invariant scalar composite operators with the highest [2,0,2] weight,

$$\begin{aligned} \mathcal{O}_1^{[2,0,2]} &= \mathrm{tr}\, z_1^2 z_2^2 - \mathrm{tr}\,(z_1 z_2)^2 = -\frac{1}{2}\mathrm{tr}\,[z_1, z_2][z_1, z_2] \\ \mathcal{O}_2^{[2,0,2]} &= 2\left(\mathrm{tr}\, z_1^2\ \mathrm{tr}\, z_2^2 - (\mathrm{tr}\, z_1 z_2)^2\right). \end{aligned} \tag{A.1}$$

The operator whose two-point functions with itself and other scalar composites receives no $\mathcal{O}(g^2)$ corrections is

$$\mathcal{Y}_{[2,0,2]} = \mathcal{O}_2^{[2,0,2]} - \frac{4}{N}\mathcal{O}_1^{[2,0,2]}. \tag{A.2}$$

[26] The dimension of an SU(N) representation with Dynkin labels $\lambda = (\lambda_1, \ldots, \lambda_{N-1})$ is $\dim|\lambda| = \Pi_{1\leq i<j\leq N} \frac{(l_i - l_j + j - i)}{(j-i)}$ where $l_i = \lambda_i + \lambda_{i+1} + \cdots + \lambda_{N-1}$. For SU(4), this becomes $\dim|[p, q, r]| = \frac{1}{12}(p+1)(q+1)(r+1)(p+q+2)(q+r+2)(p+q+r+3)$. For the large representations we are dealing with, referring to representations by their dimensions (e.g. using **84** for [2,0,2]) quickly gets cumbersome, and we keep track only of the Dynkin labels.

Other $\frac{1}{4}$-BPS operators in the same representation (the ones corresponding to different SU(3)$\times$U(1) weights) have the same coefficients in front of double and single trace operators (provided they are normalized in the same way). For example, such are (8.28)–(8.30).

A.2 [2, 1, 2]

There are also two partitions of integer $2p+q=5$ which result in linearly independent scalar composite operators

$$\begin{aligned}\mathcal{O}_1^{[2,1,2]} &= \mathrm{tr}\, z_1^3 z_2^2 - \mathrm{tr}\, z_1^2 z_2 z_1 z_2 = -\frac{1}{2}\mathrm{tr}\,[z_1,z_2][z_1^2,z_2] \\ \mathcal{O}_2^{[2,1,2]} &= \mathrm{tr}\, z_1^3\ \mathrm{tr}\, z_2^2 - 2\mathrm{tr}\, z_1 z_2\ \mathrm{tr}\, z_1^2 z_2 + \mathrm{tr}\, z_1 z_2^2\ \mathrm{tr}\, z_1^2\,. \end{aligned} \tag{A.3}$$

The single $\frac{1}{4}$-BPS operators is

$$\mathcal{Y}_{[2,1,2]} = \mathcal{O}_2^{[2,1,2]} - \frac{6}{N}\mathcal{O}_1^{[2,1,2]}\,. \tag{A.4}$$

A.3 [2, 2, 2]

$2p+q=6$ is the lowest dimension for which there are more than two linearly independent gauge invariant operators. Here we have five:

$$\mathcal{O}_1^{[2,2,2]} = \mathrm{tr}\, z_1^4 z_2^2 - \frac{2}{3}\,\mathrm{tr}\, z_1^3 z_2 z_1 z_2 - \frac{1}{3}\,\mathrm{tr}\,(z_1^2 z_2)^2 \tag{A.5}$$

$$\mathcal{O}_2^{[2,2,2]} = \mathrm{tr}\, z_1^4\ \mathrm{tr}\, z_2^2 - 2\,\mathrm{tr}\, z_1^3 z_2\ \mathrm{tr}\, z_1 z_2 + \frac{1}{3}\left(2\,\mathrm{tr}\, z_1^2 z_2^2 + \mathrm{tr}\,(z_1 z_2)^2\right)\mathrm{tr}\, z_1^2 \tag{A.6}$$

$$\mathcal{O}_3^{[2,2,2]} = \mathrm{tr}\, z_1^3\ \mathrm{tr}\, z_1 z_2^2 - (\mathrm{tr}\, z_1^2 z_2)^2 \tag{A.7}$$

$$\mathcal{O}_4^{[2,2,2]} = \frac{1}{2}\mathrm{tr}\, z_1^4\ \mathrm{tr}\, z_2^2 - \mathrm{tr}\, z_1^3 z_2\ \mathrm{tr}\, z_1 z_2 + \frac{1}{2}\left(4\,\mathrm{tr}\, z_1^2 z_2^2 - 3\,\mathrm{tr}\,(z_1 z_2)^2\right)\mathrm{tr}\, z_1^2 \tag{A.8}$$

$$\mathcal{O}_5^{[2,2,2]} = \mathrm{tr}\, z_1^2\left(\mathrm{tr}\, z_1^2\ \mathrm{tr}\, z_2^2 - (\mathrm{tr}\, z_1 z_2)^2\right). \tag{A.9}$$

The two linear combinations of operators which satisfy $\langle \mathcal{Y}\bar{\mathcal{O}}_i\rangle = 0$ for all i, are

$$\mathcal{Y}_1^{[2,2,2]} = -\frac{8N}{(N^2-4)}\mathcal{O}_1^{[2,2,2]} + \mathcal{O}_2^{[2,2,2]} + \frac{8}{3\,(N^2-4)}\left(2\mathcal{O}_3^{[2,2,2]} + \mathcal{O}_4^{[2,2,2]}\right) \tag{A.10}$$

and the one orthogonal to it (in the sense that $\langle \mathcal{Y}_1^{[2,2,2]}(x)\bar{\mathcal{Y}}_2^{[2,2,2]}(y)\rangle = 0$)

$$\begin{aligned}\mathcal{Y}_2^{[2,2,2]} = {}& \frac{144\left(N^2-4\right)\left(N^2-2\right)}{3N^6-47N^4+248N^2-192}\mathcal{O}_1^{[2,2,2]} - \frac{3N\left(N^2-7\right)\left(3N^2+8\right)}{3N^6-47N^4+248N^2-192}\mathcal{O}_2^{[2,2,2]} - \\ & -\frac{2N\left(3N^4-23N^2+104\right)}{3N^6-47N^4+248N^2-192}\left(2\mathcal{O}_3^{[2,2,2]} + \mathcal{O}_4^{[2,2,2]}\right) + \mathcal{O}_5\,. \end{aligned} \tag{A.11}$$

A.4 [3, 1, 3]

Two $[p,q,p]$ representations have $2p+q=7$. These are [3,1,3] = **960** and [2,3,2] = **1470**. In the first case, the scalar composite operators are

$$\mathcal{O}_1^{[3,1,3]} = \frac{1}{3}\mathrm{tr}\, z_1^4 z_2^3 - \frac{1}{2}\mathrm{tr}\, z_1^3 z_2 z_1 z_2^2 - \frac{1}{2}\mathrm{tr}\, z_1^3 z_2^2 z_1 z_2 +$$

$$+\frac{1}{3}\mathrm{tr}\, z_1^2 z_2 (z_1 z_2)^2 + \frac{1}{3}\mathrm{tr}\, z_1^2 z_2^2 z_1^2 z_2 \tag{A.12}$$

$$\mathcal{O}_2^{[3,1,3]} = \mathrm{tr}\, z_1^4 \;\mathrm{tr}\, z_2^3 - 3\,\mathrm{tr}\, z_1^3 z_2 \;\mathrm{tr}\, z_1 z_2^2 + \\ + \left(2\,\mathrm{tr}\, z_1^2 z_2^2 + \mathrm{tr}\,(z_1 z_2)^2\right) \mathrm{tr}\, z_1^2 z_2 - \mathrm{tr}\, z_1 z_2^3 \;\mathrm{tr}\, z_1^3 \tag{A.13}$$

$$\mathcal{O}_3^{[3,1,3]} = -\left(\mathrm{tr}\, z_1^2 z_2^3 - \mathrm{tr}\, z_1 z_2 z_1 z_2^2\right) \mathrm{tr}\, z_1^2 + \left(\mathrm{tr}\, z_1^3 z_2^2 - \mathrm{tr}\, z_1^2 z_2 z_1 z_2\right) \mathrm{tr}\, z_1 z_2 \tag{A.14}$$

$$\mathcal{O}_4^{[3,1,3]} = \mathrm{tr}\, z_1 z_2 \left(2\,\mathrm{tr}\, z_1 z_2 \;\mathrm{tr}\, z_1 z_2^2 - \mathrm{tr}\, z_2^2 \;\mathrm{tr}\, z_1^3 - 3\,\mathrm{tr}\, z_1^2 \;\mathrm{tr}\, z_1 z_2^2\right) + \\ + \mathrm{tr}\, z_1^2 \left(\mathrm{tr}\, z_2^2 \;\mathrm{tr}\, z_1^2 z_2 + \mathrm{tr}\, z_1^2 \;\mathrm{tr}\, z_2^3\right). \tag{A.15}$$

The $\mathcal{O}(g^2)$ protected operators work out to be

$$\mathcal{Y}_1^{[3,1,3]} = -\frac{12N}{N^2-2}\mathcal{O}_1^{[3,1,3]} + \mathcal{O}_2^{[3,1,3]} - \frac{5}{N^2-2}\mathcal{O}_3^{[3,1,3]} \tag{A.16}$$

$$\mathcal{Y}_2^{[3,1,3]} = \frac{96}{N^2-4}\mathcal{O}_1^{[3,1,3]} - \frac{4N}{N^2-4}\mathcal{O}_2^{[3,1,3]} + \frac{10N}{N^2-4}\mathcal{O}_3^{[3,1,3]} + \mathcal{O}_4^{[3,1,3]}. \tag{A.17}$$

A.5 $[2,3,2]$

The other $[p,q,p]$ of SU(4) with $2p+q=7$ is the [2,3,2]. Here we have seven linearly independent operators corresponding to the highest weight state:

$$\mathcal{O}_1^{[2,3,2]} = 2\,\mathrm{tr}\, z_1^5 z_2^2 - \mathrm{tr}\, z_1^4 z_2 z_1 z_2 - \mathrm{tr}\, z_1^3 z_2 z_1 z_1 z_2 \tag{A.18}$$

$$\mathcal{O}_2^{[2,3,2]} = 2\,\mathrm{tr}\, z_1^5 \;\mathrm{tr}\, z_2^2 - 4\,\mathrm{tr}\, z_1^4 z_2 \;\mathrm{tr}\, z_1 z_2 + \left(\mathrm{tr}\, z_1^3 z_2^2 + \mathrm{tr}\, z_1^2 z_2 z_1 z_2\right) \mathrm{tr}\, z_1^2 \tag{A.19}$$

$$\mathcal{O}_3^{[2,3,2]} = \mathrm{tr}\, z_1^5 \;\mathrm{tr}\, z_2^2 - 2\,\mathrm{tr}\, z_1^4 z_2 \;\mathrm{tr}\, z_1 z_2 + \left(8\,\mathrm{tr}\, z_1^3 z_2^2 - 7\,\mathrm{tr}\, z_1^2 z_2 z_1 z_2\right) \mathrm{tr}\, z_1^2 \tag{A.20}$$

$$\mathcal{O}_4^{[2,3,2]} = 3\,\mathrm{tr}\, z_1^4 \;\mathrm{tr}\, z_1 z_2^2 - 6\,\mathrm{tr}\, z_1^3 z_2 \;\mathrm{tr}\, z_1^2 z_2 + \left(2\,\mathrm{tr}\, z_1^2 z_2^2 + \mathrm{tr}\,(z_1 z_2)^2\right) \mathrm{tr}\, z_1^3 \tag{A.21}$$

$$\mathcal{O}_5^{[2,3,2]} = 3\,\mathrm{tr}\, z_1^4 \;\mathrm{tr}\, z_1 z_2^2 - 6\,\mathrm{tr}\, z_1^3 z_2 \;\mathrm{tr}\, z_1^2 z_2 + \left(7\,\mathrm{tr}\, z_1^2 z_2^2 - 4\,\mathrm{tr}\,(z_1 z_2)^2\right) \mathrm{tr}\, z_1^3 \tag{A.22}$$

$$\mathcal{O}_6^{[2,3,2]} = -8\,(\mathrm{tr}\, z_1 z_2)^2 \mathrm{tr}\, z_1^3 - 6\,\mathrm{tr}\, z_1^2 \mathrm{tr}\, z_1 z_2 \mathrm{tr}\, z_1^2 z_2 + \\ + \mathrm{tr}\, z_1^2 \left(11\,\mathrm{tr}\, z_2^2 \;\mathrm{tr}\, z_1^3 + 3\,\mathrm{tr}\, z_1^2 \;\mathrm{tr}\, z_1 z_2^2\right) \tag{A.23}$$

$$\mathcal{O}_7^{[2,3,2]} = 7\,(\mathrm{tr}\, z_1 z_2)^2 \mathrm{tr}\, z_1^3 - 6\,\mathrm{tr}\, z_1^2 \mathrm{tr}\, z_1 z_2 \mathrm{tr}\, z_1^2 z_2 + \\ + \mathrm{tr}\, z_1^2 \left(-4\,\mathrm{tr}\, z_2^2 \;\mathrm{tr}\, z_1^3 + 3\,\mathrm{tr}\, z_1^2 \;\mathrm{tr}\, z_1 z_2^2\right). \tag{A.24}$$

The $\frac{1}{4}$-BPS operators are

$$\mathcal{Y}_1^{[2,3,2]} = -\frac{10N}{N^2-7}\mathcal{O}_1^{[2,3,2]} + \mathcal{O}_2^{[2,3,2]} + \frac{2}{N^2-7}\left(\mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]} + \mathcal{O}_5^{[2,3,2]}\right) \tag{A.25}$$

$$\mathcal{Y}_2^{[2,3,2]} = -20\mathcal{O}_1^{[2,3,2]} + \frac{2\left(N^2+2\right)}{N}\mathcal{O}_2^{[2,3,2]} - \frac{2}{N}\left(\mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]}\right) + \mathcal{O}_6^{[2,3,2]} \tag{A.26}$$

$$\mathcal{Y}_3^{[2,3,2]} = 10\mathcal{O}_1^{[2,3,2]} - \frac{\left(N^2-4\right)}{N}\mathcal{O}_2^{[2,3,2]} - \frac{2}{N}\left(\mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]}\right) + \mathcal{O}_7^{[2,3,2]}. \tag{A.27}$$

(The $\mathcal{Y}_{1,2,3}^{[2,3,2]}$ are not orthogonal. Although orthonormal linear combinations are easy to find, they look rather messy and we don't list them here.)

A.6 $\mathcal{O}_{[p,q,p]}$ and $\mathcal{K}_{[p,q,p]}$ in the large-N limit

Recall that the SO(6) Young tableau for the $[p,q,p]$ of SU(4) consists of two rows (one of length $p+q$, and the other of length p). Among the possible partitions of the highest weight tableau, there are two special ones

$$\mathcal{O}_{[p,q,p]} \sim \left(\begin{array}{l} \boxed{1}\,\cdots\,\boxed{1}\,\boxed{1}\cdots\boxed{1} \\ \underbrace{\boxed{2}\,\cdots\,\boxed{2}}_{p} \quad\; q \end{array} \right), \qquad \mathcal{K}_{[p,q,p]} \sim \left(\begin{array}{l} \boxed{1}\,\cdots\,\boxed{1}\,\boxed{1}\cdots\boxed{1} \\ \underbrace{\boxed{2}\,\cdots\,\boxed{2}}_{p} \quad\; q \end{array} \right) \tag{A.28}$$

(each continuous group of boxes stands for a single trace). Explicitly, the corresponding operators are

$$\mathcal{O}_{[p,q,p]} = \sum_{k=0}^{p} \frac{(-1)^k p!}{k!(p-k)!} \operatorname{tr} \left({z_1}^{p+q-k} {z_2}^{k} \right)_s \operatorname{tr} \left({z_1}^{k} {z_2}^{p-k} \right)_s \tag{A.29}$$

$$\mathcal{K}_{[p,q,p]} = \sum_{k=0}^{p} \frac{(-1)^k p!}{k!(p-k)!} \operatorname{tr} \left[\left({z_1}^{p+q-k} {z_2}^{k} \right)_s \left({z_1}^{k} {z_2}^{p-k} \right)_s \right] \tag{A.30}$$

after projecting $\mathrm{SU}(4) \to \mathrm{SU}(3) \times \mathrm{U}(1)$ and keeping only the highest U(1)-charge pieces. Made of only z_1 and z_2, both types of operators are annihilated by four out of the sixteen Poincaré supersymmetry generators.

$\mathcal{K}_{[p,q,p]}$ is special because it is the only single trace $[p,q,p]$ operator which can be constructed out of these fields. On the other hand, $\mathcal{O}_{[p,q,p]}$ is "the most natural" double trace composite operator in this representation. We also recognize it as the free theory chiral primary from the classification of [19].

With a slight abuse of notation, we will use the same name for operators with different SU(4) weights; e.g. all $[p,q,p]$ single trace scalar composites will be referred to as $\mathcal{K}_{[p,q,p]}$, etc.

B. $[p,q,p] \otimes [r,s,r] \otimes [0,k,0]$ and BZ triangles

Tensoring irreducible representations using Young tableaux can get quite tedious. Berenstein-Zelevinsky (BZ) triangles [25] provide a powerful way to calculate the multiplicity of the scalar representation[27] in $\lambda \otimes \mu \otimes \nu$. We will discuss the construction for SU(3) and SU(4), the generalization to higher SU(N) (but not to other Lie algebras, which is not currently known) being straightforward.

For SU(3), the triangles are constructed according to the following rules:

$$\begin{array}{ccccccc} & & & m_{13} & & & \\ & & n_{12} & & l_{23} & & \\ & m_{23} & & & & m_{12} & \\ n_{13} & & l_{12} & & n_{23} & & l_{13} \end{array} \tag{B.1}$$

where the nine non-negative integers l_{ij}, m_{ij}, n_{ij} are related to the Dynkin labels (λ_1, λ_2), (μ_1, μ_2), (ν_1, ν_2) of the highest weights of the three representations by

$$\begin{array}{lll} m_{13} + n_{12} = \lambda_1 & n_{13} + l_{12} = \mu_1 & l_{13} + m_{12} = \nu_1 \\ m_{23} + n_{13} = \lambda_2 & n_{23} + l_{13} = \mu_2 & l_{23} + m_{13} = \nu_2 \,. \end{array} \tag{B.2}$$

[27] It is conventional to choose ν^* instead of ν for the third weight.

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They must further satisfy the so-called hexagon conditions

$$\begin{aligned} n_{12} + m_{23} &= m_{12} + n_{23} \\ l_{12} + m_{23} &= m_{12} + l_{23} \\ l_{12} + n_{23} &= n_{12} + l_{23}\,. \end{aligned} \tag{B.3}$$

This means that the length of opposite sides in the hexagon formed by n_{12}, l_{23}, m_{12}, n_{23}, l_{12}, and m_{23} in (B.1) are equal, the length of a segment being the sum of its two vertices.

The number of such triangles gives the multiplicity $\mathcal{N}_{\lambda\mu\nu}$; if it is not possible to construct such a triangle, ν^* does not occur in the tensor product $\lambda \otimes \mu$.

The integers in the BZ triangles have the following origin. Each pair of indices ij, $i < j$, on the labels of the triangle is related to a positive root of SU(3). For SU(N), positive roots can be written as $\epsilon_i - \epsilon_j$, $1 \le i \le j \le N$, in terms of orthonormal vectors ϵ_i in $\mathbf{R}^N$.

The triangle encodes three sums of positive roots:

$$\begin{aligned} \mu + \nu - \lambda^* &= \sum_{i<j} l_{ij}(\epsilon_i - \epsilon_j) \\ \nu + \lambda - \mu^* &= \sum_{i<j} m_{ij}(\epsilon_i - \epsilon_j) \\ \lambda + \mu - \nu^* &= \sum_{i<j} n_{ij}(\epsilon_i - \epsilon_j)\,. \end{aligned} \tag{B.4}$$

The hexagon relations (B.3) can be seen as consistency conditions for these three expansions.

For SU(4), the BZ triangles are defined in a similar way, in terms of

$$\begin{array}{ccccccccccc} & & & & & m_{14} & & & & & \\ & & & & n_{12} & & l_{34} & & & & \\ & & & m_{24} & & & & m_{13} & & & \\ & & n_{13} & & l_{23} & & n_{23} & & l_{24} & & \\ & m_{34} & & & & m_{23} & & & & m_{12} & \\ n_{14} & & l_{12} & & n_{24} & & l_{13} & & n_{34} & & l_{14} \end{array} \tag{B.5}$$

eighteen non-negative integers, related to the Dynkin labels by

$$\begin{array}{lll} m_{14} + n_{12} = \lambda_1 & n_{14} + l_{12} = \mu_1 & l_{14} + m_{12} = \nu_1 \\ m_{24} + n_{13} = \lambda_2 & n_{24} + l_{13} = \mu_2 & l_{24} + m_{13} = \nu_2 \\ m_{34} + n_{14} = \lambda_3 & n_{34} + l_{14} = \mu_3 & l_{34} + m_{14} = \nu_3\,. \end{array} \tag{B.6}$$

Furthermore, an SU(4) BZ triangle has three hexagons[28]

$$\begin{array}{lll} n_{12} + m_{24} = m_{13} + n_{23} & n_{13} + l_{23} = l_{12} + n_{24} & l_{24} + n_{23} = l_{13} + n_{34} \\ n_{12} + l_{34} = l_{23} + n_{23} & n_{13} + m_{34} = n_{24} + m_{23} & n_{23} + m_{23} = m_{12} + n_{34} \\ m_{24} + l_{23} = l_{34} + m_{13} & m_{34} + l_{12} = l_{23} + m_{23} & l_{13} + m_{23} = l_{24} + m_{12}\,. \end{array} \tag{B.7}$$

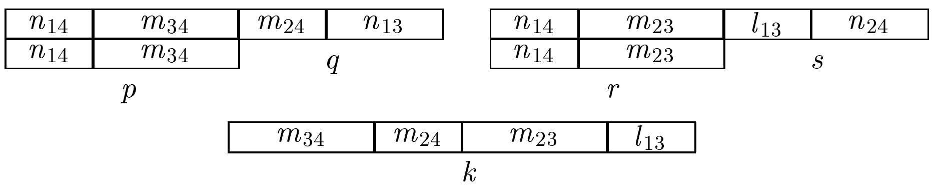
[28] The SU(N) generalization is straightforward; the BZ triangles are built out of three corner vertices and $\frac{1}{2}(N-1)(N-2)$ hexagons.

As an application, consider $\nu = [0,k,0] \subset [p,q,p] \otimes [r,s,r] = \lambda \otimes \mu$ of SU(4); here all representations are self-conjugate. The restrictions on the l_{ij}, m_{ij}, n_{ij} (these integers must all be non-negative) imply that the entries of the BZ triangle are actually

$$\begin{aligned}
m_{14} &= l_{14} = m_{12} = l_{34} = 0\,,\\
n_{12} &= p\,,\\
n_{23} &= n_{14}\,,\\
n_{34} &= r\,,\\
l_{23} &= m_{34} = p - n_{14}\,,\\
l_{12} &= m_{23} = r - n_{14}\,,\\
l_{13} &= \frac{1}{2}(s + k - (2p+q) + 2n_{14})\,,\\
m_{24} &= \frac{1}{2}(q + k - (2r+s) + 2n_{14})\,,\\
n_{13} &= \frac{1}{2}(q - k + (2r+s) - 2n_{14})\,,\\
n_{24} &= \frac{1}{2}(s - k + (2p+q) - 2n_{14})\,,\\
m_{13} &= \frac{1}{2}((2p+q) + k - (2r+s))\,,\\
l_{24} &= \frac{1}{2}((2r+s) + k - (2p+q))\,.
\end{aligned} \tag{B.8}$$

All entries thus depend on a single parameter n_{14} which is subject to restrictions $0 \leq n_{14} \leq p, r, \frac{1}{2}(p+r-k)$; plus we get further constraints $k \geq |(2p+q)-(2r+s)|$, $p+q \geq r$, and $r+s \geq p$, etc.

Now, recall that $SO(6) \sim SU(4)$, and all our operators are in fact made of the scalars ϕ^I $(I = 1, \ldots, 6)$ which are in the fundamental **6** of SO(6). In terms of the Young diagrams for SO(6), the representations involved are partitioned as



(B.9)

An especially convenient decomposition is when the $[0,k,0]$ state is made up of say only 1-s and $\bar{2}$-s. In which case, by symmetry in the vertices, there will be no contributions proportional to $\tilde{C}$ provided only two flavors are involved in the diagram. Unfortunately, this can be achieved only when $s+q \geq k$.

Alternatively, we can take a $[p,q,p]$ state made up of $n_{14}+m_{34}+m_{24}$ 1-s and $n_{14}+m_{34}+n_{13}$ 2-s; $[r,s,r]$ state made up of $n_{14}+m_{23}+l_{13}$ $\bar{1}$-s and $n_{14}+m_{23}+n_{24}$ $\bar{2}$-s; $[0,k,0]$ state made up of $m_{34}+m_{24}$ $\bar{1}$-s and $m_{23}+l_{13}$ 1-s, minus contractions. Unlike the previous decomposition, this one works for any $[0,k,0] \otimes [p,q,p] \otimes [r,s,r]$ containing the singlet.

C. Partitioning a tableau into 2 flavors

It is often convenient to choose the operators to have only two distinct flavors. Here we shall see that it can always be done.

Consider an operator in the $[p,q,p]$ of $Gl(6)$ made of n_1 1-s and n_2 2-s, to be concrete. We have the following constraints: $p \leq n_2, n_1 \leq p+q$. This state can be assigned an SU(4) weight $w = n_1(0,1,0) + n_2(1,-1,1) = (n_2, n_1 - n_2, n_2)$.

Next, we project this onto the $\mathrm{SU}(3) \times \mathrm{U}(1)$; for example, we can choose $1 \to 1 + \bar{1}$, $2 \to 2 + \bar{2}$. Then w contains terms with b 1-s, $n_1 - b$ $\bar{1}$-s, c 2-s, $n_2 - c$ $\bar{2}$-s; these have weights $w'_{b,c} = (n_2 - n_1 + 2(b-c), 2c - n_2)^{2(b+c)-(n_1+n_2)}$.

To make an irrep of $Gl(6)$ into one of SO(6), we must subtract traces. Since traces have weight zero, contributions with n contractions instead of 1 and m instead of 2, are equivalent to $n_1' = n_1 - 2n$, $n_2' = n_2 - 2m$. They are projected onto ${w'}_{b,c}^{n,m} = (n_2 - n_1 + 2(b+n) - 2(c+m), 2(c+m) - n_2)^{2(b+n+c+m)-(n_1+n_2)}$. We see that for fixed n_1 and n_2, ${w'}_{b,c}^{n,m} = {w'}_{\tilde{b},\tilde{c}}^{\tilde{n},\tilde{m}}$ iff $b + n = \tilde{b} + \tilde{n}, c + m = \tilde{c} + \tilde{m}$.

We are interested in having $b = n_1$, $c = n_2$, for example; then $w'_{n_1,n_2} = (n_1 - n_2, n_2)^{n_1+n_2}$. In order for ${w'}_{b,c}^{n,m}$ to have the same weight we must have $n_1 - n \geq b + n = n_1$, $n_2 - m \geq c + m = n_2$, or $m = n = 0$; likewise, traces also do not contribute to the projection onto $b = n_1$, $c = 0$.

This means that $[p,q,p]$ states of $\mathrm{SO}(6) \sim \mathrm{SU}(4)$ which consist of n_1 of any ϕ_a and n_2 of any other ϕ_b minus various contractions, project onto states in irreducible representations of $\mathrm{SU}(3) \times \mathrm{U}(1)$, ones containing n_1 z_a-s and n_2 z_b-s or n_1 z_a-s and n_2 $\bar{z}_b$-s, etc. *without having to subtract any traces.*

D. Details of $\left\langle \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \mathcal{O}_{\frac{1}{4}} \right\rangle$ calculations

In section 8.2 we needed to explicitly calculate several three-point functions $\langle \mathcal{O}_{[l,k,l]}(x) \mathcal{O}_{[p,q,p]}(y) \mathcal{O}_{[r,s,r]}(w) \rangle$ of $\frac{1}{4}$-BPS operators. The flavor breakdown is discussed in section 8.2, see equations (8.34)–(8.36). For the five cases of (8.32), we list the values of combinatorial coefficients multiplying the Born diagram, as well as the ones in front of $\tilde{B}(x,w)$ and $\tilde{B}(w,y)$.

When operators $\mathcal{O}_w$ are properly symmetrized, we can mark the z_1-s exchanged between $\mathcal{O}_w$ and $\mathcal{O}_x$ separately from the z_1-s exchanged between $\mathcal{O}_w$ and $\mathcal{O}_y$. As far as the combinatorial factors α_{free}, $\tilde{\beta}_{xw}$ and $\tilde{\beta}_{yw}$ are concerned, the difference is just a multiplicative factor. This would be equivalent to calculating the three-point functions with

$$\mathcal{O}_{[l,k,l]}(x) \sim [z_1^a z_2^b z_3^e] \tag{D.1}$$

$$\mathcal{O}_{[p,q,p]}(y) \sim [\bar{z}_1^c \bar{z}_2^d \bar{z}_3^e] \tag{D.2}$$

$$\mathcal{O}_{[r,s,r]}(w) \sim [\bar{z}_1^a z_1^c \bar{z}_2^b z_2^d] \tag{D.3}$$

rather than (8.34)–(8.36) instead,[29] with the same $e \equiv \frac{1}{2}[(2l+k) + (2p+q) - (2r+s)] \leq l+k, p+q$, and integers a, b, c, d partitioning $r + s = a + c$, $r = b + d$. This simplifies the calculations dramatically.

[29] As written in (D.3), $\mathcal{O}_w$ is not even a $[r,s,r]$ operator; we need to subtract SO(6) traces. But when calculating whether the three coefficients α_{free}, $\tilde{\beta}_{xw}$ and $\tilde{\beta}_{yw}$ are zero or not, the answers are the same as if we had done it properly.

For operators in the [2,0,2] or [2,1,2] representations, we chose the $\frac{1}{4}$-BPS operator from the beginning. In the other cases, several $\frac{1}{4}$-BPS chiral primaries exist in each representation, so instead we choose the operators as $\mathcal{O}_w = \sum_j C_w^j \mathcal{O}_j$ for example (see appendix A for the definitions). In all cases (D.4)–(D.17) listed above, Born level correlators are nonzero for general N, and so are the order g^2 contributions for a random set of coefficients C^j. But when we set the C^j to their proper values (to make $\mathcal{O}_y$ and $\mathcal{O}_w$ $\frac{1}{4}$-BPS), we recover correlators which are nonvanishing ($\alpha_{\rm free} \neq 0$) and protected at order g^2 ($\tilde\beta_{xw} = \tilde\beta_{yw} = 0$).[30]

With $\mathcal{O}_x$, $\mathcal{O}_y$, $\mathcal{O}_w$ as in (D.1)–(D.3), we list the representations, choices of flavors, Born level and order g^2 results below.

$$[2,0,2]\otimes[2,0,2]\otimes[2,2,2]: \qquad \langle [z_1 z_2^2 z_3]_x [\bar z_1 \bar z_2^2 \bar z_3]_y [z_1 z_2^2 \bar z_1 \bar z_2^2]_w \rangle \tag{D.4}$$

$$\begin{aligned}\alpha_{\rm free} = {} & \frac{3(N^2-1)}{2048N^3}\times \\ & \times(12288C_w^2 + 2048C_w^3 + 6144C_w^4 - 5120C_w^1 N - 4096C_w^5 N - \\ & \quad - 12800C_w^2 N^2 - 1280C_w^3 N^2 - 768C_w^4 N^2 + 2688C_w^1 N^3 - 2560C_w^5 N^3 + \\ & \quad + 3120C_w^2 N^4 + 1096C_w^3 N^4 - 168C_w^4 N^4 - 176C_w^1 N^5 + 1616C_w^5 N^5 + \\ & \quad + 8C_w^2 N^6 - 46C_w^3 N^6 - 6C_w^4 N^6 + C_w^1 N^7)\end{aligned} \tag{D.5}$$

$$\begin{aligned}\tilde\beta_{xw} = {} & \frac{9(N^2-1)(N^2-16)}{4096N^2}\times \\ & \times(768C_w^2 + 128C_w^3 + 384C_w^4 + 64C_w^1 N - 256C_w^5 N - \\ & \quad - 464C_w^2 N^2 - 216C_w^3 N^2 - 72C_w^4 N^2 - 80C_w^1 N^3 - 176C_w^5 N^3 + \\ & \quad + 8C_w^2 N^4 - 14C_w^3 N^4 - 6C_w^4 N^4 + C_w^1 N^5)\end{aligned} \tag{D.6}$$

$$\tilde\beta_{yw} = \tilde\beta_{xw}$$

$$[2,0,2]\otimes[2,1,2]\otimes[2,3,2]: \qquad \langle [z_1 z_2^2 z_3]_x [\bar z_1 \bar z_2^3 \bar z_3]_y [z_1 z_2^3 \bar z_1 \bar z_2^2]_w \rangle \tag{D.7}$$

$$\begin{aligned}\alpha_{\rm free} = {} & \frac{3(N^2-1)(N^2-4)}{10240N^4}\times \\ & \times(34560C_w^2 + 17280C_w^3 + 51840C_w^4 + 51840C_w^5 - 17280C_w^1 N - \\ & \quad - 86400C_w^6 N + 17280C_w^7 N - 24720C_w^2 N^2 + 13560C_w^3 N^2 - \\ & \quad - 11160C_w^4 N^2 + 6120C_w^5 N^2 + 6528C_w^1 N^3 + 1320C_w^6 N^3 - 3720C_w^7 N^3 + \\ & \quad + 3840C_w^2 N^4 + 6240C_w^3 N^4 + 2340C_w^4 N^4 + 1860C_w^5 N^4 - 254C_w^1 N^5 + \\ & \quad + 4140C_w^6 N^5 + 240C_w^7 N^5 + 10C_w^2 N^6 - 160C_w^3 N^6 - 10C_w^4 N^6 - \\ & \quad - 105C_w^5 N^6 + C_w^1 N^7)\end{aligned} \tag{D.8}$$

$$\begin{aligned}\tilde\beta_{xw} = {} & \frac{9(N^2-1)(N^2-4)(N^2-16)}{20480N^3}\times \\ & \times(2160C_w^2 + 1080C_w^3 + 3240C_w^4 + 3240C_w^5 - 5400C_w^6 N + \\ & \quad + 1080C_w^7 N - 960C_w^2 N^2 - 1920C_w^3 N^2 - 540C_w^4 N^2 - 1980C_w^5 N^2 -\end{aligned}$$

[30] The representations involved are quite large, so calculating Clebsch-Gordan coefficients directly for the states involved in a particular tensor product is difficult. Instead, we compute the Born level correlator and if it doesn't vanish, we know the CG is not zero. This is not a trivial check: for example if we chose the flavors as $\langle [z_1 z_2^2 z_3]_x [\bar z_1^2 \bar z_2^2 \bar z_3]_y [z_1^2 z_2^2 \bar z_1 \bar z_2^2]_w \rangle$ in (D.7), the correlator would vanish both at Born level and to order g^2.

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$$- 126C_w^1 N^3 - 660C_w^6 N^3 + 240C_w^7 N^3 + 10C_w^2 N^4 - 10C_w^4 N^4 -$$
$$- 105C_w^5 N^4 + C_w^1 N^5) \tag{D.9}$$

$$\tilde{\beta}_{yw} = \frac{3(N^2-1)(N^2-4)(N^2-36)}{5120N^3} \times$$
$$\times(1920C_w^2 + 960C_w^3 + 2880C_w^4 + 2880C_w^5 - 4800C_w^6 N +$$
$$+ 960C_w^7 N - 840C_w^2 N^2 - 1380C_w^3 N^2 - 780C_w^4 N^2 - 1740C_w^5 N^2 -$$
$$- 98C_w^1 N^3 - 540C_w^6 N^3 - 120C_w^7 N^3 + 10C_w^2 N^4 - 100C_w^3 N^4 -$$
$$- 10C_w^4 N^4 + 75C_w^5 N^4 + C_w^1 N^5) \tag{D.10}$$

$$[2,0,2] \otimes [2,1,2] \otimes [3,1,3] : \qquad \langle [z_1 z_2^2 z_3]_x [\bar{z}_1^2 \bar{z}_2^2 \bar{z}_3]_y [z_1^2 z_2^2 \bar{z}_1 \bar{z}_2^2]_w \rangle \tag{D.11}$$

$$\alpha_{\text{free}} = \frac{(N^2-1)(N^2-4)}{10240N^2} \times$$
$$\times(-103680C_w^2 + 11520C_w^1 N - 34560C_w^4 N + 18000C_w^2 N^2 - 9504C_w^3 N^2 -$$
$$- 545C_w^1 N^3 + 8520C_w^4 N^3 + 60C_w^2 N^4 + 276C_w^3 N^4 + 5C_w^1 N^5) \tag{D.12}$$

$$\tilde{\beta}_{xw} = \frac{3(N^2-1)(N^2-4)(N^2-16)}{20480N}(-2160C_w^2 + 864C_w^3 - 225C_w^1 N -$$
$$1080C_w^4 N + 60C_w^2 N^2 + 84C_w^3 N^2 + 5C_w^1 N^3)$$

$$\tilde{\beta}_{yw} = \frac{(N^2-1)(N^2-4)(N^2-36)}{10240N} \times$$
$$\times(-4320C_w^2 + 1728C_w^3 - 465C_w^1 N - 2520C_w^4 N + 60C_w^2 N^2 +$$
$$+ 228C_w^3 N^2 + 5C_w^1 N^3) \tag{D.13}$$

$$[2,0,2] \otimes [3,1,3] \otimes [2,3,2] : \qquad \langle [z_1^2 z_3^2]_x [\bar{z}_1^2 \bar{z}_2^3 \bar{z}_3^2]_y [z_1^2 z_2^3 \bar{z}_1^2]_w \rangle \tag{D.14}$$

$$\alpha_{\text{free}} = \frac{(N^2-1)(N^2-4)}{57600N^2} \times$$
$$\times(-129600C_w^2 C_y^2 - 64800C_y^2 C_w^3 - 194400C_y^2 C_w^4 - 194400C_y^2 C_w^5 + 14400C_y^1 C_w^2 N +$$
$$+ 45900C_w^1 C_y^2 N + 7200C_y^1 C_w^3 N - 8640C_w^1 C_y^3 N + 21600C_y^1 C_w^4 N -$$
$$- 43200C_w^2 C_y^4 N - 21600C_w^3 C_y^4 N - 64800C_w^4 C_y^4 N +$$
$$+ 21600C_y^1 C_w^5 N - 64800C_y^4 C_w^5 N + 324000C_y^2 C_w^6 N -$$
$$- 64800C_y^2 C_w^7 N + 88200C_w^2 C_y^2 N^2 - 3150C_y^2 C_w^3 N^2 -$$
$$- 14400C_w^2 C_y^3 N^2 + 14400C_w^3 C_y^3 N^2 + 67500C_y^2 C_w^4 N^2 -$$
$$- 12960C_y^3 C_w^4 N^2 + 18900C_w^1 C_y^4 N^2 + 43200C_y^2 C_w^5 N^2 +$$
$$+ 4320C_y^3 C_w^5 N^2 - 36000C_y^1 C_w^6 N^2 + 108000C_y^4 C_w^6 N^2 +$$
$$+ 7200C_y^1 C_w^7 N^2 - 21600C_y^4 C_w^7 N^2 - 6600C_y^1 C_w^2 N^3 -$$
$$- 13965C_w^1 C_y^2 N^3 - 9675C_y^1 C_w^3 N^3 + 3696C_w^1 C_y^3 N^3 -$$
$$- 7125C_y^1 C_w^4 N^3 + 34200C_w^2 C_y^4 N^3 - 7650C_w^3 C_y^4 N^3 +$$
$$+ 20700C_w^4 C_y^4 N^3 - 13500C_y^1 C_w^5 N^3 + 5400C_y^4 C_w^5 N^3 -$$
$$+ 4500C_y^2 C_w^6 N^3 + 31680C_y^3 C_w^6 N^3 + 11700C_y^2 C_w^7 N^3 -$$
$$- 6120C_y^3 C_w^7 N^3 - 820C_w^1 C_y^1 N^4 - 12750C_w^2 C_y^2 N^4 -$$
$$- 16050C_y^2 C_w^3 N^4 + 6420C_w^2 C_y^3 N^4 + 9930C_w^3 C_y^3 N^4 -$$

$$
\begin{aligned}
&- 10500 C_y^2 C_w^4 N^4 + 5490 C_y^3 C_w^4 N^4 - 6750 C_w^1 C_y^4 N^4 - \\
&- 13500 C_y^2 C_w^5 N^4 + 8595 C_y^3 C_w^5 N^4 - 5100 C_y^1 C_w^6 N^4 - \\
&- 9900 C_y^4 C_w^6 N^4 - 300 C_y^1 C_w^7 N^4 + 5400 C_y^4 C_w^7 N^4 + \\
&+ 50 C_y^1 C_w^2 N^5 + 210 C_w^1 C_y^2 N^5 - 925 C_y^1 C_w^3 N^5 + \\
&+ 471 C_w^1 C_y^3 N^5 + 75 C_y^1 C_w^4 N^5 - 6300 C_w^2 C_y^4 N^5 - \\
&- 10350 C_w^3 C_y^4 N^5 - 4200 C_w^4 C_y^4 N^5 + 225 C_y^1 C_w^5 N^5 - \\
&- 5850 C_y^4 C_w^5 N^5 - 12600 C_y^2 C_w^6 N^5 + 5580 C_y^3 C_w^6 N^5 + \\
&+ 180 C_y^3 C_w^7 N^5 + 5 C_w^1 C_y^1 N^6 - 30 C_w^2 C_y^3 N^6 + \\
&+ 630 C_w^3 C_y^3 N^6 + 150 C_y^2 C_w^4 N^6 + 225 C_y^2 C_w^5 N^6 - \\
&- 6300 C_y^4 C_w^6 N^6)
\end{aligned}
\tag{D.15}
$$

$$
\begin{aligned}
\tilde{\beta}_{yw} = {} & \frac{(N^2-1)(N^2-4)}{115200 N} \times \\
& \times (1036800 C_w^2 C_y^2 + 518400 C_y^2 C_w^3 - 414720 C_w^2 C_y^3 - \\
&- 207360 C_w^3 C_y^3 + 1555200 C_y^2 C_w^4 - 622080 C_y^3 C_w^4 + \\
&+ 1555200 C_y^2 C_w^5 - 622080 C_y^3 C_w^5 + 115200 C_y^1 C_w^2 N + 57600 C_y^1 C_w^3 N + \\
&+ 172800 C_y^1 C_w^4 N + 691200 C_w^2 C_y^4 N + 345600 C_w^3 C_y^4 N + 1036800 C_w^4 C_y^4 N + \\
&+ 172800 C_y^1 C_w^5 N + 1036800 C_y^4 C_w^5 N - 2592000 C_y^2 C_w^6 N + 1036800 C_y^3 C_w^6 N + \\
&+ 518400 C_y^2 C_w^7 N - 207360 C_y^3 C_w^7 N + 750 C_w^1 C_y^1 N^2 - 482400 C_w^2 C_y^2 N^2 - \\
&- 707400 C_y^2 C_w^3 N^2 + 123840 C_w^2 C_y^3 N^2 + 248400 C_w^3 C_y^3 N^2 - 516600 C_y^2 C_w^4 N^2 + \\
& 102960 C_y^3 C_w^4 N^2 + 18000 C_w^1 C_y^4 N^2 - 982800 C_y^2 C_w^5 N^2 + 289440 C_y^3 C_w^5 N^2 - \\
&- 270000 C_y^1 C_w^6 N^2 - 1296000 C_y^4 C_w^6 N^2 + 53100 C_y^1 C_w^7 N^2 + 237600 C_y^4 C_w^7 N^2 - \\
&- 52850 C_y^1 C_w^2 N^3 - 60720 C_w^1 C_y^2 N^3 - 76800 C_y^1 C_w^3 N^3 + 22488 C_w^1 C_y^3 N^3 - \\
&- 56200 C_y^1 C_w^4 N^3 - 303600 C_w^2 C_y^4 N^3 - 428400 C_w^3 C_y^4 N^3 - \\
&- 315600 C_w^4 C_y^4 N^3 - 105825 C_y^1 C_w^5 N^3 - 574200 C_y^4 C_w^5 N^3 - \\
&- 392400 C_y^2 C_w^6 N^3 + 286560 C_y^3 C_w^6 N^3 - 18000 C_y^2 C_w^7 N^3 - \\
&- 16560 C_y^3 C_w^7 N^3 - 6755 C_w^1 C_y^1 N^4 + 3000 C_w^2 C_y^2 N^4 - \\
&- 70200 C_y^2 C_w^3 N^4 + 29160 C_w^2 C_y^3 N^4 + 70920 C_w^3 C_y^3 N^4 + \\
&+ 4800 C_y^2 C_w^4 N^4 + 29640 C_y^3 C_w^4 N^4 - 40680 C_w^1 C_y^4 N^4 + \\
&+ 13500 C_y^2 C_w^5 N^4 + 52020 C_y^3 C_w^5 N^4 - 45600 C_y^1 C_w^6 N^4 - \\
&- 309600 C_y^4 C_w^6 N^4 - 1500 C_y^1 C_w^7 N^4 + 250 C_y^1 C_w^2 N^5 + \\
&+ 300 C_w^1 C_y^2 N^5 - 8000 C_y^1 C_w^3 N^5 + 3948 C_w^1 C_y^3 N^5 + \\
&+ 400 C_y^1 C_w^4 N^5 - 51600 C_w^3 C_y^4 N^5 + 1125 C_y^1 C_w^5 N^5 + \\
&+ 30960 C_y^3 C_w^6 N^5 + 25 C_w^1 C_y^1 N^6 + 5160 C_w^3 C_y^3 N^6)
\end{aligned}
\tag{D.16}
$$

$$
[2,0,2] \otimes [3,1,3] \otimes [3,1,3] : \qquad \langle [z_1^2 z_3^2]_x [\bar{z}_1^2 \bar{z}_2^3 \bar{z}_3^2]_y [z_1^2 z_2^3 \bar{z}_1^2]_w \rangle
\tag{D.17}
$$

$$
\alpha_{\text{free}} = \frac{(N^2-1)(N^2-4)}{1036800 N^2} \times
$$

$$
\begin{aligned}
&\times(2332800C_w^2C_y^2 - 2332800C_y^2C_w^4N - 2332800C_w^2C_y^4N + 10800C_w^1C_y^1N^2 + \\
&\quad + 324000C_w^2C_y^2N^2 - 492480C_y^2C_w^3N^2 - 492480C_w^2C_y^3N^2 + 114048C_w^3C_y^3N^2 + \\
&\quad + 259200C_y^1C_w^4N^2 + 259200C_w^1C_y^4N^2 - 777600C_w^4C_y^4N^2 - \\
&\quad - 44100C_y^1C_w^2N^3 - 44100C_w^1C_y^2N^3 - 8280C_y^1C_w^3N^3 - \\
&\quad - 8280C_w^1C_y^3N^3 + 950400C_y^2C_w^4N^3 - 250560C_y^3C_w^4N^3 + \\
&\quad + 950400C_w^2C_y^4N^3 - 250560C_w^3C_y^4N^3 - 5875C_w^1C_y^1N^4 - \\
&\quad - 133200C_w^2C_y^2N^4 + 90720C_y^2C_w^3N^4 + 90720C_w^2C_y^3N^4 - \\
&\quad - 22752C_w^3C_y^3N^4 - 52800C_y^1C_w^4N^4 - 52800C_w^1C_y^4N^4 + \\
&\quad + 396000C_w^4C_y^4N^4 + 900C_y^1C_w^2N^5 + 900C_w^1C_y^2N^5 + \\
&\quad + 4920C_y^1C_w^3N^5 + 4920C_w^1C_y^3N^5 - 100800C_y^2C_w^4N^5 + \\
&\quad + 51840C_y^3C_w^4N^5 - 100800C_w^2C_y^4N^5 + 51840C_w^3C_y^4N^5 + \\
&\quad + 75C_w^1C_y^1N^6 + 3600C_w^2C_y^2N^6 - 2880C_w^3C_y^3N^6 - -50400C_w^4C_y^4N^6)
\end{aligned} \tag{D.18}
$$

$$
\begin{aligned}
\tilde{\beta}_{yw} = {} & \frac{(N^2-1)(N^2-4)}{518400} \times \\
&\times(172800C_y^1C_w^2 + 172800C_w^1C_y^2 - 69120C_y^1C_w^3 - 69120C_w^1C_y^3 + \\
&\quad + 4147200C_y^2C_w^4 - 1658880C_y^3C_w^4 + 4147200C_w^2C_y^4 - 1658880C_w^3C_y^4 + \\
&\quad + 33450C_w^1C_y^1N - 837000C_w^2C_y^2N - 79920C_y^2C_w^3N - \\
&\quad - 79920C_w^2C_y^3N + 197856C_w^3C_y^3N + 457200C_y^1C_w^4N + \\
&\quad + 457200C_w^1C_y^4N + 2678400C_w^4C_y^4N - 107700C_y^1C_w^2N^2 - \\
&\quad - 107700C_w^1C_y^2N^2 - 2640C_y^1C_w^3N^2 - 2640C_w^1C_y^3N^2 - \\
&\quad - 910800C_y^2C_w^4N^2 + 96480C_y^3C_w^4N^2 - 910800C_w^2C_y^4N^2 + \\
&\quad + 96480C_w^3C_y^4N^2 - 13175C_w^1C_y^1N^3 + 18000C_w^2C_y^2N^3 + \\
&\quad + 83880C_y^2C_w^3N^3 + 83880C_w^2C_y^3N^3 - 43200C_w^3C_y^3N^3 - \\
&\quad - 100800C_y^1C_w^4N^3 - 100800C_w^1C_y^4N^3 - 597600C_w^4C_y^4N^3 + \\
&\quad + 1500C_y^1C_w^2N^4 + 1500C_w^1C_y^2N^4 + 9480C_y^1C_w^3N^4 + \\
&\quad + 9480C_w^1C_y^3N^4 + 59760C_y^3C_w^4N^4 + 59760C_w^3C_y^4N^4 + \\
&\quad + 125C_w^1C_y^1N^5 - 5976C_w^3C_y^3N^5)\,.
\end{aligned} \tag{D.19}
$$

The last two sets of calculations are very formidable. Getting the coefficients $\tilde{\beta}_{yw}$ took 2430 hrs for $[2,0,2] \otimes [3,1,3] \otimes [3,1,3]$ and 2830 hrs for $[2,0,2] \otimes [3,1,3] \otimes [2,3,2]$ of a *Mathematica* computation on a Pentium-III with 1.4MHz speed.

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